Priority Queues and Binary Heaps

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In this lecture, we will learn our first “tree data structure”—called the binary heap—which serves as an implementation of the priority queue.
A priority queue stores a set $S$ of $n$ integers and supports the following operations:

- **Insert($e$):** Adds a new integer to $S$.
- **Delete-min:** Removes the smallest integer in $S$, and returns it.
Example

Suppose that the following integers are inserted into an initially empty priority queue: 93, 39, 1, 26, 8, 23, 79, 54.

If next we perform a Delete-Min, the operation returns 1, after which $S = \{93, 39, 26, 8, 23, 79, 54\}$.

The next Delete-Min returns 8, and leaves $S = \{93, 39, 26, 23, 79, 54\}$.

Unlike an ordinary queue (which obeys FIFO), a priority queue guarantees that the elements always leave in ascending order, regardless of the order by which they are inserted.
Next we will implement a priority queue using a data structure called the binary heap to achieve the following guarantees:

- $O(n)$ space consumption
- $O(\log n)$ insertion time
- $O(\log n)$ delete-min time.
Let $S$ be a set of $n$ integers. A **binary heap** on $S$ is a binary tree $T$ satisfying:

1. $T$ is complete.
2. Every node $u$ in $T$ corresponds to a distinct integer in $S$—the integer is called the **key** of $u$ (and is stored at $u$).
3. If $u$ is an internal node, the key of $u$ is smaller than those of its child nodes.

**Note:**

- Condition 2 implies that $T$ has $n$ nodes.
- Condition 3 implies that the key of $u$ is the smallest in the subtree of $u$. 
Two possible binary heaps on $S = \{93, 39, 1, 26, 8, 23, 79, 54\}$:

![Binary Heaps Diagram]

The smallest integer of $S$ must be the key of the root.
We perform $\text{insert}(e)$ on a binary heap $T$ as follows:

1. Create a leaf node $z$ with key $e$, while ensuring that $T$ is a complete binary tree—notice that there is only one place where $z$ can be added.

2. Set $u \leftarrow z$.

3. If $u$ is the root, return.

4. If the key of $u >$ the key of its parent $p$, return.

5. Otherwise, swap the keys of $u$ and $p$. Set $u \leftarrow p$, and repeat from Step 3.
Example

Assume that we want to insert 15 into the binary heap below:
First, add 15 as a new leaf, making sure that we still have a complete binary tree.

15 causes a violation by being smaller than its parent. This is fixed by a swap with its parent; see next.
15 still causes a violation, necessitating another swap, as shown next.
Example

No more violation. Insertion complete.
Delete-Min

We perform a delete-min on a binary heap $T$ as follows:

1. Report the key of the root.
2. Identify the rightmost leaf $z$ at the bottom level of $T$.
3. Delete $z$, and store the key of $z$ at the root.
4. Set $u \leftarrow$ the root.
5. If $u$ is a leaf, return.
6. If the key of $u <$ the keys of the children of $u$, return.
7. Otherwise, let $v$ be the child of $u$ with a smaller key. Swap the keys of $u$ and $v$. Set $u \leftarrow v$, and repeat from Step 5.
Example

Assume that we perform a delete-min from the binary heap below:
Example

First, find the rightmost leaf at the bottom level, namely, 79.

Notice that the tree is still a complete binary tree after removing this leaf.
Example

Remove the leaf, but place the value 79 in the root.

79 causes a violation by being greater than its children. This is fixed by swapping it with node 8, which is the child of the root with a smaller key. See the next slide.
Node 79 still has a violation, causing another swap as shown next.
The final tree after the delete-min.
How to Find the Rightmost Leaf at the Bottom Level

Before analyzing the running time of insert and delete-min, let us first consider a sub-problem:

Given a complete binary tree $T$ with $n$ nodes, how to identify quickly the rightmost leaf node at the bottom level of $T$.

Our aforementioned algorithms depend on a fast solution to the above.
Next, we give a clever algorithm for solving the sub-problem in $O(\log n)$ time.

First, write the value of $n$ in binary form.

Skip the most significant bit. We will scan the remaining bits from left to right, and descend as instructed by the next bit:

- If the next bit is 0, we go to the left child of the current node.
- Otherwise, go to the right child.
Here $n = 9$, which is 1001 in binary. Skipping the first bit 1, we scan the remaining bits and descend accordingly:

- The 2nd leftmost bit is 0; so we turn left, and go to node 15.
- The 3rd leftmost bit is 0; so we turn left, and go to node 39.
- The 4th leftmost bit is 1; so we turn right, and go to node 79 (done).
Analysis of Insertion and Delete-Min

We are now ready to prove that our insertion and delete-Min algorithms finish in $O(\log n)$ time.

It suffices to point out the key facts:

- Step 1 of the insertion algorithm (Slide 8) and Step 2 of the delete-min algorithm (Slide 13) can be performed in $O(\log n)$ time, using our solution to the previous sub-problem.

- The rest of insertion ascends a root-to-leaf path, while that of delete-min descends a root-to-leaf path. The time is $O(\log n)$ in both cases.
Now officially we have reached the following conclusion. We can maintain a priority queue on a set $S$ of elements such that:

- At any moment, the space consumption is $O(n)$, where $n = |S|$.
- An insertion can be processed in $O(\log n)$ time.
- A delete-min can be processed in $O(\log n)$ time.