Graphs: Basic Concepts

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This will be a simple lecture that paves the foundation for our subsequent discussion on graphs. We will define a series of concepts on undirected/directed graphs, and discuss how they can be represented in a computer.
Recall that we have defined undirected graphs in an earlier lecture.

**Undirected Graphs**

An **undirected graph** is a pair of \((V, E)\) where:

- \(V\) is a set of elements, each of which called a **node**.
- \(E\) is a set of pairs \((u, v)\) such that:
  - \(u\) and \(v\) are **distinct** nodes;
  - If \((u, v)\) is in \(E\), then \((v, u)\) is also in \(E\)—we say that there is an **edge** between \(u\) and \(v\).

A node may also be called a **vertex**. We will refer to \(V\) as the **vertex set** or the **node set** of the graph, and \(E\) the **edge set**.
This is an undirected graph where there are 5 vertices $v_1, v_2, ..., v_5$, and 5 edges $e_1, e_2, ..., e_5$. 
An directed graph is a pair of \((V, E)\) where:

- \(V\) is a set of elements, each of which called a node.
- \(E\) is a set of pairs \((u, v)\) where \(u\) and \(v\) are nodes in \(V\). We say that there is a (directed) edge from \(u\) to \(v\).

A node may also be called a vertex. We will refer to \(V\) as the vertex set or the node set of the graph, and \(E\) the edge set.

A (directed) edge \((u, v)\) is said to be an outgoing edge of \(u\), and an incoming edge of \(v\). Accordingly, \(v\) is an out-neighbor of \(u\), and \(u\) an in-neighbor of \(v\).
This is an directed graph \((V, E)\) where there are 5 vertices \(v_1, v_2, \ldots, v_5\), and 7 edges \(e_1, e_2, \ldots, e_7\). Note that every edge has a direction. Edge \(e_6\), for instance, is an outgoing edge of \(v_5\), and an incoming edge of \(v_4\).
In an undirected graph, the degree of a vertex $u$ is the number of edges of $u$.

In a directed graph, the out-degree of a vertex $u$ is the number of outgoing edges of $u$, and its in-degree is the number of its incoming edges.

Example

In the left graph, the degree of $v_5$ is 2. In the right graph, the out-degree of $v_3$ is 2, and its in-degree is 1.
Next, we discuss two common ways to store a graph: adjacency list and adjacency matrix. In both cases, we represent each vertex in $V$ using a unique id in $1, 2, \ldots, |V|$. 
Adjacency List – Undirected Graphs

Each vertex \( u \in V \) is associated with a linked list that enumerates all the vertices that are connected to \( u \).

Example 1.

Space = \( O(|V| + |E|) \).
Adjacency List – Directed Graphs

Each vertex $u \in V$ is associated with a linked list that enumerates all the vertices $v \in V$ such that there is an edge from $u$ to $v$.

**Example 2.**

![Diagram showing a directed graph and its adjacency list representation]

$v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5$

$e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6 \quad e_7$

Space = $O(|V| + |E|)$. 
Adjacency Matrix – Undirected Graphs

A \(|V| \times |V|\) matrix \(A\) where \(A[u, v] = 1\) if \((u, v) \in E\), or 0 otherwise.

Example 3.

\[
\begin{array}{c|ccccc}
& v_1 & v_2 & v_3 & v_4 & v_5 \\
\hline
v_1 & 0 & 1 & 1 & 1 & 1 \\
v_2 & 1 & 0 & 0 & 0 & 0 \\
v_3 & 1 & 0 & 0 & 0 & 0 \\
v_4 & 1 & 0 & 0 & 0 & 1 \\
v_5 & 1 & 0 & 0 & 1 & 0
\end{array}
\]

- \(A\) must be symmetric.
- Space = \(O(|V|^2)\).

**Think:** How to store \(A\) so that, for any vertices \(u, v \in V\), we can find out if they have an edge in constant time?
Adjacency Matrix – Directed Graphs

Defined in the same way as in the undirected case.

Example 4.

\[
\begin{array}{c|cccccc}
& v_1 & v_2 & v_3 & v_4 & v_5 \\
\hline
v_1 & 0 & 1 & 0 & 1 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 1 & 0 & 1 & 0 & 0 \\
v_4 & 0 & 0 & 0 & 0 & 1 \\
v_5 & 1 & 0 & 0 & 1 & 0 \\
\end{array}
\]

- A may not be symmetric.
- Space = $O(|V|^2)$. 