Problem 1 (Correctness of Dijkstra) Prove that Dijkstra’s algorithm correctly computes all the shortest paths from the source vertex.

Solution. Let $s$ be the source vertex. Recall that the algorithm works by repetitively removing the vertex $u$ from $S$ that has the smallest $\text{dist}(u)$. We will prove that, when $u$ is removed, $\text{dist}(u)$ equals precisely the shortest path distance—denoted as $\text{spdist}(u)$—from $s$ to $u$.

We will prove the claim by induction on the sequence of vertices removed. This is obviously true for the first vertex removed, which is $s$ itself with $\text{dist}(s) = 0$.

Now consider that we are removing vertex $u$ from $S$, and the claim is true with respect to all the vertices already removed. Consider any shortest path $\pi$ from $s$ to $u$. Let $v$ be the predecessor of $u$ on this path. We will prove that $v$ has already been removed. This will complete the proof because when $v$ is removed, we have:

- $\text{spdist}(v) = \text{dist}(v)$
- Relaxing the edge $(v, u)$ makes $\text{dist}(u) = \text{dist}(v) + w(u, v) = \text{spdist}(v)$.

We will prove that all the vertices on $\pi$ have been removed (and hence, $v$ as well) at the moment when $u$ is removed. Suppose that this is not true. Let $v'$ be the first vertex (in the direction from $s$ to $u$) on $\pi$ that still remains in $S$. Let $p$ be the predecessor of $v'$ on $\pi$. By the inductive assumption, we know that $\text{dist}(p) = \text{spdist}(p)$ when $p$ was removed. Hence, after relaxing the edge $(p, v')$, we had $\text{dist}(v') = \text{dist}(p) + w(p, v') = \text{spdist}(v') < \text{dist}(u)$. This means that $v'$ should be the next vertex to remove, contradicting that the algorithm has chosen $u$.

Problem 2. Let $S$ be a set of integer pairs of the form $(id, v)$. We will refer to the first field as the $id$ of the pair, and the second as the $key$ of the pair. Design a data structure that supports the following operations:

- Insert: add a new pair $(id, v)$ to $S$ (you can assume that $S$ does not already have a pair with the same id).
- Delete: given an integer $t$, delete the pair $(id, v)$ from $S$ where $t = id$, if such a pair exists.
- DeleteMin: remove from $S$ the pair with the smallest key, and return it.

Your structure must consume $O(n)$ space, and support all operations in $O(\log n)$ time where $n = |S|$.

Solution. Maintain $S$ in two binary search trees $T_1$ and $T_2$, where the pairs are indexed on ids in $T_1$, and on keys in $T_2$. We support the three operations as follows:

- Insert: simply insert the new pair $(id, v)$ into both $T_1$ and $T_2$.
- Delete: first find the pair with id $t$ in $T_1$, from which we know the key $v$ of the pair. Now, delete the pair $(t, v)$ from both $T_1$ and $T_2$.
- DeleteMin: find the pair with the smallest key $v$ from $T_2$ (which can be found by continuously descending into left child nodes). Now we have its id $t$ as well. Remove $(t, v)$ from $T_1$ and $T_2$. 

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Problem 3. Describe how to implement the Dijkstra’s algorithm on a graph $G = (V,E)$ in $O((|V| + |E|) \cdot \log |V|)$ time.

Solution. Recall that the algorithm maintains (i) a set $S$ of vertices at all times, and (ii) an integer value $\text{dist}(v)$ for each vertex $v \in S$. Define $P$ to be the set of $(v, \text{dist}(v))$ pairs (one for each $v \in S$). We need the following operations on $P$:

- Insert: add a pair $(v, \text{dist}(v))$ to $P$.
- DecreaseKey: given a vertex $v \in S$ and an integer $x < \text{dist}(v)$, update the pair $(v, \text{dist}(v))$ to $(v, x)$ (and thereby, setting $\text{dist}(v) = x$ in $P$).
- DeleteMin: Remove from $P$ the pair $(v, \text{dist}(v))$ with the smallest $\text{dist}(v)$.

We can store $P$ in a data structure of Problem 2 which supports all operations in $O(\log |V|)$ time (note: DecreaseKey can be implemented as a Delete followed by an Insert).

In addition to the above structure, we store all the $\text{dist}(v)$ values in an array $A$ of length $|V|$, so that using the id of a vertex $v$, we can find its $\text{dist}(v)$ in constant time.

Now we can implement the algorithm as follows. Initially, insert only $(s, 0)$ into $P$, where $s$ is the source vertex. Also, in $A$, set all the values to $\infty$, except the cell of $s$ which equals 0.

Then, we repeat the following until $P$ is empty:

- Perform a DeleteMin to obtain a pair $(v, \text{dist}(v))$.
- For every edge $(v, u)$, compare $\text{dist}(u)$ to $\text{dist}(v) + w(u,v)$. If the latter is smaller, perform a DecreaseKey on vertex $u$ to set $\text{dist}(u) = \text{dist}(v) + w(u,v)$, and update the cell of $u$ in $A$ with this value as well.

Problem 4. Prove: in a weighted undirected graph $G = (V,E)$ where all the edges have distinct weights, the minimum spanning tree (MST) is unique.

Solution. We will prove that the tree $T$ returned by the Prim’s algorithm is the only MST. Set $n = |V|$. Let $e_1, e_2, ..., e_{n-1}$ be the sequence of edges that the algorithm adds to $T$. Suppose, on the contrary, that there is another MST $T'$. Let $k$ be the smallest $i$ such that $e_i$ is not in $T'$.

- Case 1: $k = 1$. This means that $e_1$, which is the edge with the smallest weight, is not in $T'$. Add $e_1$ to $T'$ to create a cycle, and remove from the cycle the edge with the largest weight. This create another spanning tree whose cost is strictly smaller than $T'$ (remember: all the edges are distinct), contradicting the fact that $T'$ is an MST.

- Case 2: $k > 1$. Recall that edges $e_1, e_2, ..., e_{k-1}$ form a tree. Let $S$ be the set of vertices in this tree. Add $e_k = \{u,v\}$ into $T'$ to create a cycle. Suppose $u \in S$; it follows that $v \notin S$. Let us walk on the cycle from $v$, by going into $S$, traveling within $S$, and stopping as soon as we exist $S$. Let $\{u',v'\}$ be the last edge crossed (namely, one of $u', v'$ is in $S$, while the other one is not). By the way Prim’s algorithm runs and the fact that all edges have distinct weights, we know that $\{u, v\}$ has a smaller weight than $\{u', v'\}$. Thus, removing $\{u', v'\}$ from $T'$ gives spanning tree with strictly smaller cost, which creates a contradiction.

Problem 5. Describe how to implement the Prim’s algorithm on a graph $G = (V,E)$ in $O((|V| + |E|) \cdot \log |V|)$ time.
Solution. Remember that the algorithm incrementally grows a tree $T$ which at the end becomes the final minimum spanning tree. Let $S$ be the set of vertices that are currently in $T$. At all times, the algorithm maintains, for every vertex $v \in V \setminus S$, its lightest extension edge $\text{best-ext}(v)$, and the weight of this edge.

To implement this, we maintain a set $P$ of triples, one for every vertex $v \in V \setminus S$. Specifically, the triple of $v$ has the form $(v, u, t)$, indicating that $\text{best-ext}(v)$ is the edge $(u, v)$, whose weight is $t$. We need the following operations on $P$:

- **Insert**: add a triple $(v, u, t)$ to $P$.
- **DecreaseKey**: given a vertex $v \in S$ and an extension edge $(v, u')$, this operation does the following. First, fetch the triple $(v, u, t)$. Then, compare $t$ to the weight $t'$ of $(v, u')$. If $t' < t$, update the triple $(v, u, t)$ to $(v, u', t')$; otherwise, do nothing.
- **DeleteMin**: Remove from $P$ the triple $(v, u, t)$ with the smallest $t$.

We can store $P$ in a data structure of Problem 2 which supports all operations in $O(\log |V|)$ time (note: DecreaseKey can be implemented as a Delete followed by an Insert). Besides the above structure, we also store an array $A$ of length $|V|$ to so that we can query in constant time, for any vertex $v \in V$, whether $v$ is in $S$ currently.

Now we can implement the algorithm as follows. Let $\{u_1, u_2\}$ be an edge with the smallest weight in $G$. The set $S$ contains only $u_1$ and $u_2$ at this point. For every vertex $v \in V \setminus S$ where $S = \{u_1, u_2\}$, we check whether $v$ has extension edges to $u_1$ and $u_2$. If neither edge exists, insert triple $(v, \text{nil}, \infty)$ to $P$. Otherwise, suppose without loss of generality that $(v, u_1)$ is the lighter extension edge of $v$ with weight $t$ (if $v$ has only one extension edge, then $(v, u_1)$ is this edge). We insert a triple $(v, u_1, t)$ into $P$.

Repeat the following until $P$ is empty:

- Perform a DeleteMin to obtain a triple $(v, u, t)$.
- Recall that $v$ should be added to $S$, which may need to change the extension edges of some other vertices. To implement this, for every edge $(v, u)$ of $v$ where $u \notin S$, perform DecreaseKey with $v$ and $(v, u)$. 
