Problem 1 (Correctness of the White Path Theorem) Consider performing DFS on a directed graph $G = (V, E)$. Then, both of the following statements are true:

- Suppose that when a vertex $u$ is discovered, there is still a white path from $u$ to a vertex $v$ (namely, we can hop from $u$ to $v$ while stepping on only white vertices). Then, $v$ must be a descendant of $u$ in the DFS forest.
- Conversely, if $v$ is a descendant of $u$ in the DFS forest, then there must be a white path from $u$ to $v$ at the moment when $u$ is discovered.

Solution.

Proof of the First Statement. Let $\pi$ be the path from $u$ to $v$. We will prove that all the vertices on $\pi$ must be descendants of $u$ in the DFS forest. Suppose that this is not true. Let $v'$ be the first vertex on $\pi$—in the order from $u$ to $v$—that is not a descendant of $u$. Clearly, $v' \neq u$. Let $u'$ be the vertex that precedes $v'$ on $\pi$.

Consider the moment before $u'$ turns black. As $u'$ is a descendant of $u$ in the DFS forest, we know that $u$ is in the stack currently. The color of $v'$ cannot be white—otherwise, DFS must now push $v'$ into the stack, which is a contradiction of the fact that $u'$ is turning black. On the other hand, if $v'$ is either gray or black, it means that $v$ must have been pushed into the stack while $u$ still remains in the stack. This contradicts the fact that $v$ is not a descendant of $u$.

Proof of the Second Statement. As $v$ is a descendant of $u$, there is a moment in DFS when $u$ and $v$ were both in the stack with $v$ being the top of the stack. It thus follows that there is a white path from $u$ to $v$ when $u$ is discovered.

Problem 2 (DFS on Undirected Graphs). Let $G = (V, E)$ be an undirected graph. Consider the execution of DFS on $G$. The algorithm runs in exactly the same way as DFS on a directed graph. The only difference is that, a vertex $u$ is popped out of the stack, only if none of its neighbors (instead of out-neighbors) is still white. Give a possible DFS tree produced if we (i) start DFS on $a$ in the following graph, and (ii) follow the convention that we explore the neighbors of a vertex in alphabetic order.

![Diagram](image)
Problem 3 (No Cross Edges in Undirected DFS). Let $G = (V, E)$ be an undirected graph. Consider the DFS forest produced by running DFS on $G$ (assuming arbitrary starting and restarting vertices). Let $\{u, v\}$ be an edge in $G$ (note that we use the notation $\{u, v\}$, instead of $(u, v)$, to emphasize that the edge has no directions). Prove: either $u$ is an ancestor of $v$, or $v$ is an ancestor of $u$.

Remark: Because of this lemma, we can classify each edge $\{u, v\}$ in $G$ as follows:

- **Tree edge**: if $u$ is the parent of $v$ or $v$ is the parent of $u$.
- **Back edge**: otherwise.

Solution. The white path theorem—as stated in Problem 1—still holds for undirected DFS (the same proof applies here as well). Between $u$ and $v$, let $u$ be the vertex discovered first. Then, the white path theorem says that $v$ must be a descendant of $u$.

Problem 4 (Undirected Cycle Detection). Let $G = (V, E)$ be an undirected graph. A cycle is a sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{t-1}, v_t\}$ where $v_t = v_1$. Adapt DFS to design an algorithm to detect whether $G$ has a cycle in $O(|V| + |E|)$ time.

Solution. Perform DFS on $G$. Declare cycle presence if and only if a back edge is found. For example, in the Solution of Problem 2, there is such an edge $\{a, d\}$, which implies a cycle.

Problem 5** (Articulation Vertex). Let $G = (V, E)$ be an undirected graph that is connected (i.e., there is a path between any two distinct vertices). A vertex $u \in V$ is called an articulation vertex if the following is true: $G$ becomes disconnected after removing $u$ and all the edges of $u$. For example, in the figure below, vertex $g$ is an articulation, and so is $d$. No other vertices are articulation vertices.

Consider any DFS tree on $G$. Prove:

- If a vertex $u$ is a leaf in the DFS tree, it cannot be an articulation vertex.
- Let $u$ a vertex that is neither a leaf in the DFS tree nor the root. It is an articulation vertex if and only if the following is true:
– There is at least one child \( v \) of \( u \), such that no back edge connects a descendant of \( v \) to a proper ancestor of \( u \).

- Let \( u \) be the root of a DFS tree. It is an articulation vertex if and only if it has at least two child nodes in the DFS tree.

**Solution.**

**Proof of the First Bullet.**

Suppose that \( u \) is an articulation vertex. Let \( s \) be the starting vertex of the DFS. Then there must be a vertex \( u' \) such that all the paths from \( s \) to \( u' \) must go by way of \( u \). This implies that, when \( v \) is discovered by DFS, there must be a white path from \( u \) to \( u' \). The white path theorem then says that \( u' \) must be a descendant of \( u \), contradicting the fact that \( u \) is a leaf.

**Proof of the Second Bullet.**

**Only-if direction.** Imagine removing \( u \) from \( G \), which should disconnect \( G \). Let \( C_1, C_2, \ldots, C_t \) for some \( t \geq 2 \) be the connected components (CCs) of the resulting graph (recall that a CC is a set of vertices that are reachable from each other). Without loss of generality, assume that \( s \) belongs to \( C_1 \). Consider the moment right before the first vertex \( v \) in \( C_2 \) is discovered. It must be a child of \( u \) in the DFS tree (because any path from \( s \) to \( u \) must cross the edge \( \{u, v\} \)). At this moment, all the vertices in \( C_2 \) must be white; and they are the only vertices that \( v \) can reach via white paths. Hence, all the vertices of \( C_2 \) must be the *only* descendants of \( v \). It thus follows that there can be no back edge connecting a descendant of \( v \) to a proper ancestor of \( u \).

**If direction.** We will prove that, after \( u \) is removed from \( G \), \( s \) can no longer reach \( v \), which thus indicates that \( u \) is an articulation vertex. Suppose, on the contrary, that \( u \) can still access \( v \) by a path \( \pi \) (that does not contain \( u \)). Denote the vertices on \( \pi \) as \( v_1, v_2, \ldots, v_x \) with \( v_1 = s \) and \( v_x = v \). Let \( v_i \) (for some \( i \in [1, x] \)) be the last vertex on \( \pi \) that is an ancestor of \( u \). We will prove that \( v_{i+1} \) must be a descendant of \( v \), making \( \{v_i, v_{i+1}\} \) a back edge that connects a descendant of \( v \) to a proper ancestor of \( u \), which contradicts the fact that no such back edges exist.

Consider the moment right before the discovery of \( v \). We argue that the colors of \( v_{i+1}, v_{i+2}, \ldots, v_x \) must all be white at this moment:

- First, none of them can be gray—otherwise, such a vertex must be an ancestor of \( u \) (because \( u \) is the parent of \( v \)), contradicting the definition of \( v_i \).

- If \( v_{i+1} \) is black, it means that \( v_{i+1} \) was discovered before \( v \). Furthermore, when \( v_{i+1} \) turned black, \( v_{i+2} \) cannot be white (otherwise, DFS would have crossed the edge \( \{v_{i+1}, v_{i+2}\} \) to push \( v_{i+2} \) into the stack). Thus, at this moment, \( v_{i+2} \) must be black (as mentioned, \( v_{i+2} \) cannot be gray currently). Following the same argument, we obtain that \( v_{i+3}, v_{i+4}, \ldots, v_x \) must all be black at the moment. However, this contradicts the fact that \( v_x = v \) is white.

- The same argument proves that none of \( v_{i+2}, v_{i+3}, \ldots, v_x \) can be black.

Therefore, all of \( v_{i+1}, v_{i+2}, \ldots, v_x \) must be descendants of \( v \).

**Proof of the Third Bullet.**

**Only-if direction.** Vertex \( u \) is the starting vertex of DFS. Imagine removing \( u \) from \( G \), which should disconnect \( G \) into CCs \( C_1, C_2, \ldots, C_t \) for some \( t \geq 2 \). Let \( v \) be the second vertex discovered by DFS (i.e., right after \( u \)). Without loss of generality, suppose that \( v \in C_1 \). Then, when \( v \) is discovered,
there is no white path from $v$ to any vertex in $C_2$. Hence, none of the vertices in $C_2$ can be descendants of $v$, implying that $u$ must have another child.

If direction. Let $v$ be the second vertex discovered by DFS (i.e., right after $u$). Let $v'$ any other child of $u$ in the DFS tree. We will prove that any path from $v$ to $v'$ must go through $u$, which indicates that $u$ is an articulation vertex.

Assume that there is a path $\pi$ from $v$ to $v'$ that does not go through $u$. Then, when $v$ is discovered, there is a white path from $v$ to $v'$, which means that $v'$ must be a descendant of $v$ in the DFS tree. This contradicts the fact that $v'$ and $v$ are siblings.

**Problem 6* (Finding an Articulation Vertex).** Let $G = (V, E)$ be an undirected graph that is connected. Design an algorithm to determine whether $G$ has any articulation vertex. Your algorithm must finish in $O(|V| + |E|)$ time.

**Solution.** First grow a DFS-tree $T$, but make sure that at each node $u$ we record its level (the root is at level 0), denoted as $\text{level}(u)$. We now process the vertices of $T$ in a bottom-up manner (i.e., descending order of level). Let $u$ be a vertex to be processed next. We do the following:

- **Case 1: $u$ is a leaf node:** We inspect all the edges $\{u, v\}$ of $u$, and obtain:
  
  $$\text{highest-back-level}(u) = \min_{\{u, v\}} \text{level}(v).$$

- **Case 2: $u$ is an internal node but not the root:** Let $v_1, v_2, \ldots, v_t$ be its children (which have already been processed). If
  
  $$\min_{i=1}^{t} \text{highest-back-level}(v_i) \geq \text{level}(u)$$

we report $u$ as an articulation vertex, and finish.

Otherwise, inspect all the edges $\{u, v\}$ of $u$, and obtain:

$$\text{highest-back-level}(u) = \min_{\{u, v\}} \text{level}(v).$$

Then, update $\text{highest-back-level}(u)$ to be:

$$\min \left\{ \text{highest-back-level}(u), \min_{i=1}^{t} \text{highest-back-level}(v_i) \right\}.$$  

- **Case 3: $u$ is the root:** Report $u$ as an articulation vertex if it has at least 2 child nodes.

**Problem 7 (The $L$-Ordering Lemma of the SCC Algorithm).** Prove the lemma on Slide 28 of the lecture notes about strongly connected components (SCCs). Let $S_1, S_2$ be SCCs such that there is a path from $S_1$ to $S_2$ in $G^{\text{SCC}}$. In the ordering of $L$, the earliest vertex in $S_2$ must come before the earliest vertex in $S_1$.

**Solution.** All the notations in this proof follow those defined in the lecture notes. Let $X_1, X_2, \ldots, X_t$ be a path on $G^{\text{SCC}}$ such that $X_1 = S_1$ and $X_t = S_2$. Consider the DFS performed on the reverse graph $G^R$. Let $v$ be the first vertex discovered among all the vertices of $X_1 \cup X_2 \cup \ldots \cup X_t$ in this DFS. Assume without loss of generality that $v \in S_i$ for some $i \in [1, t]$. 

Observe that regardless of the value of $i$, at the moment right before the discovery of $v$, there is a white path in $G^R$ from $v$ to all the vertices in $X_1$. In other words, all the vertices in $X_1$ must turn black before $v$ in the DFS. It thus follows that all of them must turn black before the last vertex $u$ of $S_2$ that turns black. Therefore, $u$ is behind all the vertices of $S_1$ in $L^R$, which indicates that $u$ is before all the vertices of $S_1$ in $L$.

**Problem 8.** Prove that for any directed graph $G = (V,E)$, the SCC decomposition is unique. Namely, there is only one way to decompose $V$ into disjoint subsets, each of which is an SCC; and furthermore, such a decomposition always exists.

**Solution.** First, we prove that every vertex $v$ must belong to some SCC $S \subseteq V$. We can construct $S$ as follows. Initially, $S = \{v\}$. Repeat the following:

1. Enumerate all vertices in $u \in V \setminus S$ to see if
   - $u$ can reach all vertices in $S$, and
   - every vertex in $S$ can reach $u$.

2. If $u$ does not exist, done; $S$ is an SCC containing $v$.

3. Otherwise, add $u$ to $S$, and repeat from Step 1.

Second, we prove that, for every vertex $v \in V$, there is a unique SCC containing $v$. This follows directly from the fact that no two distinct SCCs can have non-empty intersection (we proved this fact during the class).

We now conclude that the SCC decomposition is unique.