Problem 1. Consider the kernel function $K(p, q) = (p \cdot q + 1)^3$, where $p = (p[1], p[2])$ and $q = (q[1], q[2])$ are 2-dimensional vectors. Recall that there is a mapping function $\phi$ from $\mathbb{R}^2$ to $\mathbb{R}^d$ for some integer $d$, such that $K(p, q)$ equals the dot product of $\phi(p)$ and $\phi(q)$. Give the details of $\phi$.

Answer: Rewrite $K$ as dot product form.


Problem 2. Consider a set $P$ of 2D points each colored either red or blue. It is known that the red and blue points can be separated by applying the Kernel function $K(p, q) = (p \cdot q + 1)^2$. Prove that they can also be separated by the kernel function $K'(p, q) = (2p \cdot q + 3)^2$.

Answer: Using the method explained in Problem 1, we can find the mapping functions $\phi$ and $\phi'$ for $K$ and $K'$, respectively:


Let $\pi$ be the plane that separates the red and blue points under $\phi$. If $w \cdot \phi(x) = 0$ is the equation for $\pi$, then (i) for every red point $p$, $w \cdot \phi(p) > 0$, and (ii) for every blue point $p$, $w \cdot \phi(p) < 0$.

Set $w' = (\frac{w[1]}{2}, \frac{w[2]}{2}, \frac{w[3]}{3}, \frac{w[4]}{\sqrt{6}}, \frac{w[5]}{\sqrt{6}}, \frac{w[6]}{\sqrt{6}})$. Let $\pi'$ be the plane given by the equation $w' \cdot \phi(x) = 0$. We claim that $\pi'$ also separates the red and blue points. Indeed, for every red point $p$, we have:

$$w' \cdot \phi'(p)$$

$$= \frac{w[1]}{2} \cdot 2p[1]^2 + \frac{w[2]}{2} \cdot 2p[2]^2 + \frac{w[3]}{3} \cdot 3 + \frac{w[4]}{\sqrt{6}} \cdot 2\sqrt{3}p[1] + \frac{w[5]}{\sqrt{6}} \cdot 2\sqrt{3}p[2] + \frac{w[6]}{\sqrt{6}} \cdot 2\sqrt{3}p[1]p[2]$$


$$= w \cdot \phi(x) > 0.$$

Likewise, we can prove that, for every blue point $p$, it holds that $w' \cdot \phi'(p) = w \cdot \phi(p) < 0$.

Problem 3. Consider a set $P$ of 2D points that has 3 red points $p_1(-2, -2), p_2(1, 1), p_3(3, 3)$, and 2 blue points $q_1(-2, 2), q_2(2, -2)$. Suppose that we use Perceptron to find a separation plane $\pi$ using the Kernel function $K(x, y) = (x \cdot y + 1)^2$. Then, according to $\pi$, what is the color of point $(2, 2)$?

Answer: Initially, let $c_0 = 0$. Perceptron runs as follows:
Iteration 1. Since $c_0 \cdot \phi(p_1) = 0$, we set $c_1 = c_0 + \phi(p_1) = \phi(p_1)$.

Iteration 2. Since $c_1 \cdot \phi(q_1) = K(p_1, q_1) = 1 > 0$, we set $c_2 = c_1 - \phi(q_1) = \phi(p_1) - \phi(q_1)$.

Iteration 3. There are no more violations for $c_2$. So we have found a separation plane $c_2 \cdot \phi(x) = 0$ such that (i) $c_2 \cdot \phi(x) > 0$ for every red point $p$, and (ii) $c_2 \cdot \phi(x) < 0$ for every blue point $p$.

Now consider the point $r = (2, 2)$. As $c_2 \cdot \phi(r) = K(p_1, r) - K(q_1, r) = 48 > 0$, we classify $r$ as red.

**Problem 4.** Same settings as in Problem 3. Calculate the distance from $\phi(p_1)$ to the separation plane you find in the feature space.

**Answer:** We know from the solution of Problem 3 that the weight vector of the separation plane (in the feature space) is $w = \phi(p_1) - \phi(q_1)$.

The distance from $\phi(p_1)$ to this plane equals

$$
\frac{w \cdot \phi(p_1)}{|w|} = \frac{w \cdot \phi(p_1)}{\sqrt{w \cdot w}} = \frac{(\phi(p_1) - \phi(q_1)) \cdot \phi(p_1)}{\sqrt{(\phi(p_1) - \phi(q_1)) \cdot (\phi(p_1) - \phi(q_1))}}
$$

$$
= \frac{\phi(p_1) \cdot \phi(p_1) - \phi(p_1) \cdot \phi(q_1)}{\sqrt{\phi(p_1) \cdot \phi(p_1) - 2\phi(p_1) \cdot \phi(q_1) + \phi(q_1) \cdot \phi(q_1)}}
$$

$$
= \frac{K(p_1, p_1) - K(p_1, q_1)}{\sqrt{K(p_1, p_1) - 2K(p_1, q_1) + K(q_1, q_1)}}
$$

$$
= \frac{81 - 1}{\sqrt{81 - 2 \times 1 + 81}}
$$

$$
= \frac{80}{\sqrt{160}}.
$$