Lecture Notes: Legality Means Delaunay

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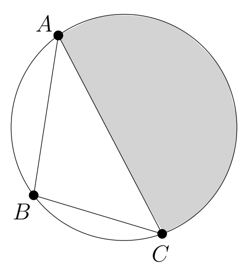
This document is supplementary to the main lecture notes and provides an alternative proof for:

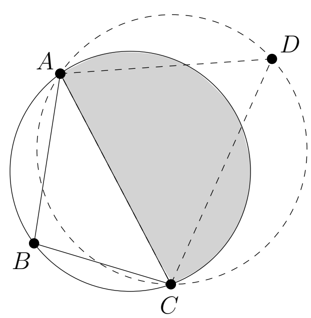
Theorem 1. *Let P be a set of 2D points in general position (no four points are on the same circle). If G is a legal triangulation of P, then G must be the Delaunay triangulation of P.*

If a triangle has vertices A, B, and C, we denote the triangle as ΔABC. Consider its circumcircle $\odot ABC$. The points A, B, and C divide the circle into three arcs: $\widehat{AC}$, $\widehat{CB}$, and $\widehat{BA}$. We define the *territory* of $\widehat{AC}$ inside $\odot ABC$ as the region bounded by segment $\overline{AC}$ and arc $\widehat{AC}$ (see the gray area in the figure below).



We will need the rudimentary geometric fact below (proof left as an exercise):

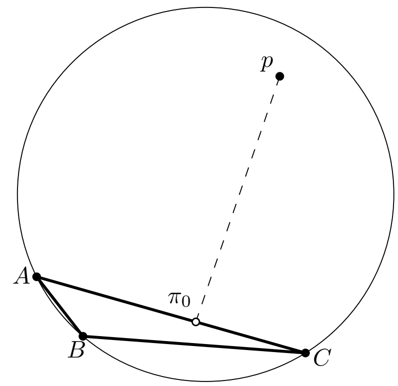
Lemma 2. *Let D be a point outside $\odot ABC$ such that points B and D fall on different sides of the line passing through segment $\overline{AC}$. Then, $\odot ACD$ covers the territory of arc $\widehat{AC}$ inside $\odot ABC$ (see the figure below for an illustration).*



To prove Theorem 1, it suffices to establish the following claim:

Claim: The circumcircle of each triangle in G contains no points of P in the interior.

Assume, for contradiction, that the circumcircle of some triangle Δ in G contains a point $p \in P$ in the interior. Name the vertices of Δ as A, B, and C in such way that points p and B fall on different sides of the line passing AC. Such naming is possible because p does not fall inside Δ (recall that G is a triangulation of P). Identify an arbitrary point π_0 in the interior of segment $\overline{AC}$. W.l.o.g., we consider that no point of P lies in the interior of segment $\overline{\pi_0 p}$ (otherwise, redefine p to that point).



Shoot a ray ρ from from π_0 to p. The ray passes through a sequence of triangles $\Delta_0, \Delta_1, ..., \Delta_t$ satisfying:

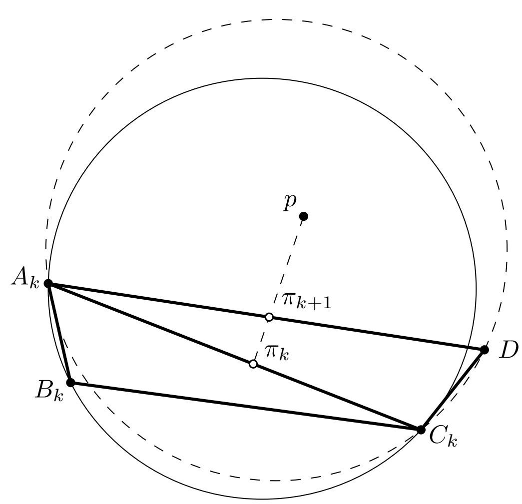
- $\Delta_0 = \Delta ABC$;
- Δ_i and Δ_{i+1} share an edge for each $i \in [0, t-1]$;
- p is a vertex of Δ_t.

We will show that the edge shared by Δ_{t-1} and Δ_t must be illegal, thus contradicting the fact that G is a legal triangulation. This will validate the claim and, hence, Theorem 1.

For each $i \in [0, t-1]$, define π_i as the point where ρ exits Δ_i (this is also the point where ρ enters Δ_{i+1}).

Lemma 3. *For each $i \in [0, t-1]$, the circumcircle of Δ_i — denoted as $\odot_i$ — covers the segment $\overline{\pi_i p}$.*

Proof. We will prove the lemma by induction. Its correction for $i = 0$ follows directly from the definition of π_0 and p. Assuming that the lemma holds for $i = k \in [0, t-2]$, next we will prove its correctness for $i = k + 1$.



Name the vertices of Δ_k as A_k, B_k and C_k in such a way that points B_k and p fall on different sides of the line passing segment $\overline{A_k C_k}$. By the inductive assumption, circle $\odot_k$ covers segment $\overline{\pi_k p}$. This means that $\overline{\pi_k p}$ falls in the territory of segment $\overline{A_k C_k}$ in $\odot_k$. See the figure above for an illustration.

Points A_k and C_k must be two vertices of Δ_{k+1}; denote by D the remaining vertex of Δ_{k+1}. By the fact that $\overline{A_k C_k}$ is a legal edge, point D must fall outside $\odot_k$. Lemma 2 then assures us that circle $\odot_{k+1}$ (the circumcircle of $\Delta A_k C_k D$) covers the territory of segment $\overline{A_k C_k}$ in $\odot_k$. This further implies that $\odot_{k+1}$ must cover the entire segment $\overline{\pi_k p}$ and, hence, the segment $\overline{\pi_{k+1} p}$. □

We now know that $\odot_{t-1}$ covers segment $\overline{\pi_{t-1} p}$. As p cannot be on (the boundary of) $\odot_{t-1}$ (no four points co-circular), p must be in the interior of $\odot_{t-1}$. By this contradicts the fact that the edge shared by Δ_{t-1} and Δ_t is legal.