Lecture Notes: Minimum Enclosing Balls

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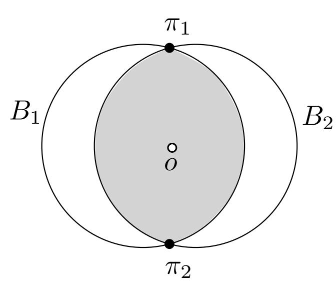
Let P be a set of n points in $\mathbb{R}^d$. We want to find a ball B with the smallest radius to cover all the points in P. We refer to B as the *minimum enclosing ball* (MEB) of P and denote it as $\text{MEB}(P)$. The MEB of P can be found in $O(n)$ expected time in any constant dimensionality. This lecture will explain how to do so in 2D space, and in an exercise you will be asked to extend the algorithm to $d \geq 3$. Our discussion will make the general position assumption that no four points fall on the same circle.

1 Geometric Facts in 2D Space

Lemma 1. *There is only one ball with the smallest radius covering all the points in P.*

Proof. Assume, on the contrary, that there are two such balls B_1 and B_2; see the figure below. Then, P must be covered by the shaded area. Let π_1 and π_2 the intersection points of the two balls. Consider the ball B centering at the midpoint o of the segment $\overline{\pi_1\pi_2}$ and having a radius half the length of $\overline{\pi_1\pi_2}$. The ball B covers the shaded area (and hence, also P) but is smaller than B_1 and B_2, giving a contradiction.

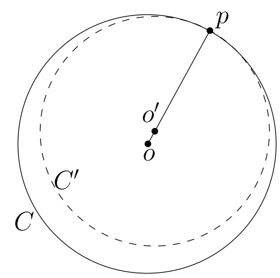


□

Lemma 2. *The boundary of* $\text{MEB}(P)$ *passes at least two points of P.*

Proof. Let C be the boundary of $\text{MEB}(P)$. If C passes no points of P, shrink C infinitesimally to obtain a smaller ball still covering P, which contradicts the definition of C.

Suppose that C passes only one point $p \in P$. Let o be the center of C. Consider sliding a point o' from o towards p infinitesimally, and look at the circle C' centered at o' with radius equal to the length of segment $\overline{o'p}$. C' is smaller than C but still contains P in the interior. This again gives a contradiction.



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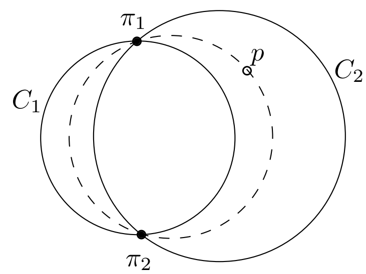
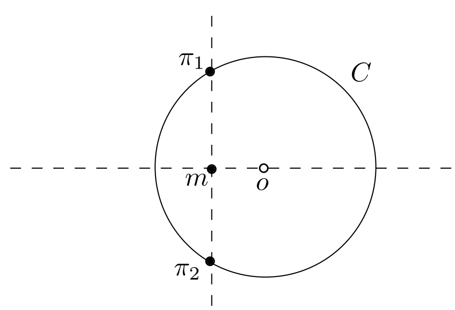
Lemma 3. *Let C_1 and C_2 be two intersecting circles such that C_1 is no larger than C_2 (in terms of radius). Denote by L the area inside both circles. Consider an arbitrary point p that is covered by C_2 but not by C_1. Then, there exists a circle C that is smaller than C_2, passes points p, π_1, π_2, and covers the area L.*

See the figure below for an illustration, where C is the circle in dashed line.



Proof. Let us first review a rudimentary geometric fact. Fix two distinct points π_1 and π_2. Consider all the circles passing both π_1 and π_2. The centers of these circles must be on the perpendicular bisector of segment $\overline{\pi_1\pi_2}$. Every such circle C can be divided into (i) a left arc, which is the part of C on the left of $\overline{\pi_1\pi_2}$, and (ii) a right arc, which is the part of C on the right of $\overline{\pi_1\pi_2}$. As the center o of C moves away from the midpoint m of segment $\overline{\pi_1\pi_2}$ towards right, the left arc "morphs" towards $\overline{\pi_1\pi_2}$, while the right arc "morphs" away from $\overline{\pi_1\pi_2}$; furthermore, C grows continuously. The behavior is symmetric when o moves towards left.



Returning to the context of the lemma, let π_1 and π_2 be the intersection points of C_1 and C_2. Imagine "morphing" a circle C from C_2 to C_1 while ensuring that C passes π_1 and π_2. Stop as soon as the right arc of C hits p. As C_1 is no larger than C_2, we know that C must be smaller than C_2 (think: why?). Thus, C is the circle we are looking for. ∎

2 Two Points Are Known

Next, we will discuss a variant of the MEB problem. Let p_1 and p_2 be two points in P such that at least one ball has the following property: it encloses the entire P and its boundary passes both p_1 and p_2. We want to find such a ball with the smallest radius, denoted as $\text{MEB}(P, \{p_1, p_2\})$; this ball must be unique (the proof is similar to that of Lemma 1 and left as an exercise). We can solve the problem in $O(n)$ time using the algorithm below.

Algorithm 1: Two-Points-Fixed-MEB(P, $\{p_1, p_2\}$)

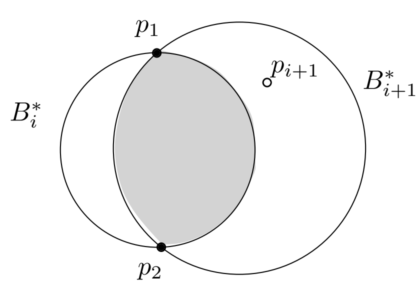
$B \leftarrow$ the smallest ball covering p_1 and p_2
$(p_3, p_4, ..., p_n) \leftarrow$ an arbitrary permutation of the other points in P
for $i = 3$ **to** n **do**
    if $p_i \notin B$ **then**
        $B \leftarrow$ the ball whose boundary passes p_1, p_2, and p_i
return B

The next lemma proves the algorithm's correctness.

Lemma 4. *Define $P_i = \{p_1, ..., p_i\}$ for each $i \in [1, n]$. For any $i \in [2, n]$, define $B_i^* = \text{MEB}(P_i, \{p_1, p_2\})$. If $p_{i+1} \in B_i^*$, then $B_{i+1}^* = B_i^*$. Otherwise, the boundary of B_{i+1}^* must pass p_{i+1}.*

Proof. If $p_{i+1} \in B_i^*$, then $B_{i+1}^* = B_i^*$ follows from the uniqueness of $\text{MEB}(P_{i+1}, \{p_1, p_2\})$. Next, we consider $p_{i+1} \notin B_i^*$.

Assume on the contrary that the boundary of B_{i+1}^* does not pass p_{i+1}. Hence, p_{i+1} falls in the interior B_{i+1}^*. The radius of B_{i+1}^* cannot be smaller than that of B_i^* (both of them cover P_i and pass p_1 and p_2, but B_i^* is $\text{MEB}(P_i, \{p_1, p_2\})$). The entire P_i must fall in the intersection of B_i^* and B_{i+1}^* (the shaded area in the figure below). By Lemma 3, there exists a ball smaller than B_{i+1}^* covering P_{i+1} and passing p_1, p_2, which gives a contradiction. □



3 One Point Is Known

Next, we will look at a less restrictive variant of the problem. Let p_1 be a point in P such that at least one ball has the following property: it encloses the entire P and its boundary passes p_1. We want to find such a ball with the smallest radius, denoted as $\text{MEB}(P, \{p_1\})$; this ball must be unique (the proof is similar to that of Lemma 1 and left as an exercise). We can solve the problem using the algorithm below.

Algorithm 2: One-Point-Fixed-MEB(P, $\{p_1\}$)

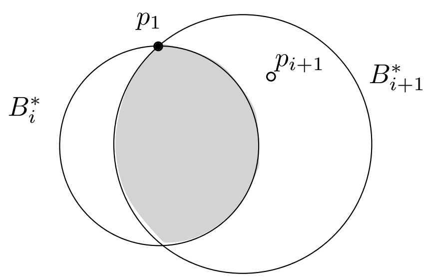
1 $(p_2, p_3, ..., p_n) \leftarrow$ a random permutation of $P \setminus \{p_1\}$
2 $B \leftarrow$ the smallest ball covering p_1 and p_2
3 **for** $i = 3$ **to** n **do**
4   **if** $p_i \notin B$ **then**
5         $B \leftarrow$ Two-Points-Fixed-MEB($\{p_1, ..., p_i\}$, $\{p_1, p_i\}$)
6 **return** B

The next lemma proves the algorithm's correctness.

Lemma 5. *Define $P_i = \{p_1, ..., p_i\}$ for each $i \in [1, n]$. For any $i \in [2, n]$, define $B_i^* = \text{MEB}(P_i, \{p_1\})$. If $p_{i+1} \in B_i^*$, then $B_{i+1}^* = B_i^*$. Otherwise, the boundary of B_{i+1}^* must pass p_{i+1}.*

Proof. The argument is nearly identical to the one used to prove Lemma 4. We will focus only one the case where $p_{i+1} \notin B_i^*$.

Assume on the contrary that the boundary of B_{i+1}^* does not pass p_{i+1}. Hence, p_{i+1} falls in the interior B_{i+1}^*. The radius of B_{i+1}^* cannot be smaller than that of B_i^*. The entire P_i must fall in the intersection of B_i^* and B_{i+1}^* (the shaded area in the figure below). By Lemma 3, there exists a ball smaller than B_{i+1}^* covering P_{i+1} and passing p_1, which gives a contradiction.



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Let us analyze the running time of the algorithm. Let t_i be the expected time of the iteration (Lines 3-5) for a specific $i \in [3, n]$. We will prove $\mathbf{E}[t_i] = O(1)$. At the beginning of the iteration, $B = B_{i-1}^*$ (guaranteed by the above lemma). The iteration takes $O(i)$ time if $p_i \notin B_{i-1}^*$, or $O(1)$ time otherwise.

Other than p_1, the boundary of B_{i-1}^* must pass at least one more point in P_i (the proof is similar to that of Lemma 2 and left to you), but no more than two more points (due to our general position assumption). We deal with these cases separately:

- B_{i-1}^* passes two more points $\pi_1, \pi_2 \in P_i$. The event $p_i \notin B_{i-1}^*$ occurs only if $p_i = \pi_1$ or $p_i = \pi_2$, which happens with probability $2/(i-1)$ (backward analysis).

- B passes only one more point $\pi_1 \in P_i$. The event $p_i \notin B_{i-1}^*$ occurs only if $p_i = \pi_1$, which happens with probability $1/(i-1)$ (backward analysis).

It thus follows that $\mathbf{E}[t_i] = O(1)$.

4 No Point Is Known

We are ready to tackle the MEB problem in its most general form.

Algorithm 3: MEB(P)

$(p_1, ..., p_n) \leftarrow$ a random permutation of P
$B \leftarrow$ the smallest ball covering p_1 and p_2
for $i = 3$ **to** n **do**
  if $p_i \notin B$ **then**
    $B \leftarrow$ One-Point-Fixed-MEB($\{p_1, ..., p_i\}$, $\{p_i\}$)
return B

The next lemma proves the algorithm's correctness.

Lemma 6. *Define* $P_i = \{p_1, ..., p_i\}$ *for each* $i \in [1, n]$*. For any* $i \in [2, n]$*, define* $B_i^* = \text{MEB}(P_i)$*. If* $p_{i+1} \in B_i^*$*, then* $B_{i+1}^* = B_i^*$*. Otherwise, the boundary of* B_{i+1}^* *must pass* p_{i+1}*.*

Proof. The argument is again nearly identical to the one used to prove Lemma 4. We will discuss only the case where $p_{i+1} \in B_i^*$. Assume on the contrary that the boundary of B_{i+1}^* does not pass p_{i+1}. Hence, p_{i+1} falls in the interior B_{i+1}^*. The radius of B_{i+1}^* cannot be smaller than that of B_i^*. The entire P_i must fall in the intersection of B_i^* and B_{i+1}^*. By Lemma 3, there exists a ball smaller than B_{i+1}^* covering P_{i+1}, which gives a contradiction. □

We can once again apply backward analysis to prove that Algorithm 3 runs in $O(n)$ expected time. The details should have become straightforward and are left as an exercise.