Exercises for CSCI5010

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Problem 1. Let P be a set of n points in $\mathbb{R}^d$. Define d_{max} as the maximum distance between two points in P, namely, $d_{max} = \max_{p,q \in P} dist(p,q)$, where $dist(p,q)$ is the Euclidean distance between p and q. Describe an $O(n)$-time algorithm to find a d-dimensional box such that:

- the box has the same side length on each dimension, and the side length is $\Theta(d_{max})$;
- the box covers all the points in P.

Problem 2. Let P be a set of n points in $\mathbb{R}^d$. Define d_{max} as the maximum distance between two points in P, namely, $d_{max} = \max_{p,q \in P} dist(p,q)$. Define d_{min} as the minimum distance between two distinct points in P, namely, $d_{min} = \min_{\text{distinct } p,q \in P} dist(p,q)$. Describe how to build a quadtree on P in $O(n \cdot \log \frac{d_{max}}{d_{min}})$ time.

Problem 3. Let P be a set of points in $\mathbb{R}^d$. Suppose that we have constructed an s-well separated point decomposition for P: $\{\{A_1, B_1\}, \{A_2, B_2\}, ..., \{A_m, B_m\}\}$. Let $\{p, q\}$ be a closest pair of P (i.e., $dist(p,q) = d_{min}$, where d_{min} is as defined in Problem 2). Prove: if $s > 2$, then there exists an $i \in [1, m]$ such that A_i contains only p, and B_i contains only q.

Hint 1: Recall that there must exist an $i \in [1, m]$ such that $p \in A_i$ and $q \in B_i$.

Hint 2: If you do not want to think, read Lect 18 of Prof. Mount's notes.

Problem 4*. Let P be a set of points in $\mathbb{R}^d$. Suppose that we have constructed an s-well separated point decomposition for P: $\{\{A_1, B_1\}, \{A_2, B_2\}, ..., \{A_m, B_m\}\}$. For each $i \in [1, m]$, let a_i be an arbitrary point in A_i, and b_i be an arbitrary point in B_i. Let us construct an undirected graph $G = (V, E)$ as follows:

- $V = P$, namely, each vertex of V is a point in P.
- For each $i \in [1, m]$, add to E an edge $\{a_i, b_i\}$.

Prove: If $s > 2$, then G must be connected (i.e., for any two points $p, q \in P$, G has a path from p to q).

Hint: Imagine listing all distinct pairs of the points in P in ascending order of distance. Apply induction on the sorted list.