Exercises for CSCI 5010

Prepared by Yufei Tao

Problem 1. P is a set of points in $\mathbb{R}^2$. Prove: if we take a point p from P uniformly at random, the number of Voronoi neighbors of p is $O(1)$ in expectation.

Problem 2*. Let P be a set of points in $\mathbb{R}^2$. Consider any point q in $\mathbb{R}^2$ (which may not be in P). Let p_1 be the nearest neighbor of q, and p_2 be the second nearest neighbor of q (i.e., p_2 has the second smallest distance to q among all the points in P). Prove: p_2 must be a Voronoi neighbor of p_1.

Hint: Argue for the existence of a circle that passes p_1, p_2 and contains no points of P in the interior.

Problem 3.** Let P be a set of n points in $\mathbb{R}^2$. Describe a data structure of $O(n)$ space that can find the two nearest neighbors of any query point $q \in \mathbb{R}^2$ in $O(\log n)$ time.

Hint: This problem is rather challenging because there are many details to sort out. It is probably suited only for the students that want to research theoretical computer science. Consider the Voronoi cell of any point $p \in P$. Think about building a tiny Voronoi diagram inside that Voronoi cell. To build this tiny diagram, it suffices to consider only the Voronoi neighbors of p. The challenging part is to argue that all the tiny Voronoi diagrams have $O(n)$ complexity.

Spoiler: If you have given up, Google for "order-2 Voronoi diagram".

Problem 4 (Reading Exercise). Prove: Every triangulation of a set P of n points contains $2n - 2 - h$ triangles and $3n - 3 - h$ edges, where h is the number of points on the convex hull boundary of P.

Hint: Apply Euler's formula, but if you do not want to think, read Section 9.1 of the textbook.

Problem 5 (Textbook Exercise 9.13). The *Garibel graph* of a set P of points in $\mathbb{R}^2$ is a graph G defined as follows. The vertex set of G is P, i.e., each point of P is a vertex in G. Two points p and q are connected by an edge in G if and only if the circle with segment $\overline{pq}$ as a diameter contains no points of P in the interior. Prove:

- Every edge of G is in the Delaunay triangulation of P.
- Two points p and q are connected in G if and only if the segment $\overline{pq}$ intersects with the Voronoi edge shared by the Voronoi cells of p and q.

Problem 6*. Let P be a set of n points in $\mathbb{R}^2$. Suppose that you are given the Delaunay triangulaiton of P in the adjacency format (i.e., for each point $p \in P$, you have a linked list containing the neighbors of P in the triangulation). Describe an algorithm to compute the convex hull of P in $O(n)$ time.

Hint: How can you tell whether a point $p \in P$ is on the boundary of the convex hull from its neighbors? Also be reminded that you are supposed to output the vertices of the convex hull in an appropriate order (e.g., clockwise).