Dynamic Programming: Piggyback, Dependency, and Space

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Principle of Dynamic Programming

- Remember the output of every subproblem to avoid re-computation.
- Resolve subproblems according to an appropriate order.

Problem 2 (Regular List 6)

In the lecture, we derived for the rod cutting problem:

$$opt(n) \quad = \quad \max_{i=1}^{n}(P[i] + opt(n-i)).$$

Define *bestSub*$(n) = k$ if the above maximization is obtained at $i = k$.

Example

length i	1	2	3	4
price $P[i]$	1	5	8	9
$opt(i)$	1	5	8	10
$bestSub(i)$	1	2	3	2

How to compute $bestSub(1), bestSub(2), ..., bestSub(n)$ in $O(n^2)$ time?

Solution

First, compute $opt(1), opt(2), ..., opt(n)$ in $O(n^2)$ time, as discussed in the lecture.

For each $t \in [1, n]$, compute $bestSub(t)$ as follows:

- Identify the $k \in [1, t]$ maximizing $P[k] + opt(t - k)$.
    - This takes $O(t)$ time.
- Set $bestSub(t) = k$.

Doing so for all $t \in [1, n]$ takes $O(n^2)$ time.

The idea of computing $bestSub(t)$ for all $t \in [1, n]$ is called the **piggyback technique**.

Problem 2 (cont.)

In the lecture, we derived for the rod cutting problem:

$$opt(n) = \max_{i=1}^{n}(P[i] + opt(n-i)).$$

Define $bestSub(n) = k$ if the above maximization is obtained at $i = k$.

Suppose that we have already computed $bestSub(1)$, $bestSub(2)$, ..., $bestSub(n)$. How do we output an optimal cutting method — namely, a sequence of lengths achieving the maximum revenue — in $O(n)$ time?

Solution

1. $\ell \leftarrow n$
2. **while** $\ell > 0$ **do**
3. output "length $bestSub(\ell)$"
4. $\ell \leftarrow \ell - bestSub(\ell)$

Example

length i	1	2	3	4
price $P[i]$	1	5	8	9
$opt(i)$	1	5	8	10
$bestSub(i)$	1	2	3	2

Output:
length 2
length 2

Problem 3 (Regular List 6)

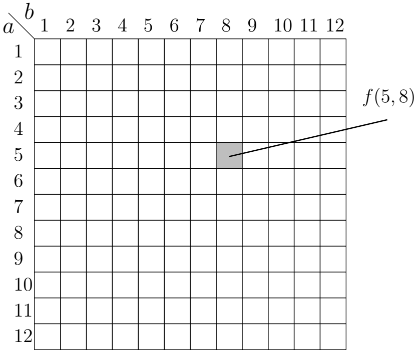
Let A be an array of n integers. Define function $f(a, b)$ — where $a \in [1, n]$ and $b \in [1, n]$ — as follows:

$$f(a,b) = \begin{cases} 0 & \text{if } a \geq b \\ \left(\sum_{c=a}^{b} A[c]\right) + \min_{c=a+1}^{b-1}\{f(a,c) + f(c,b)\} & \text{otherwise} \end{cases}$$

Design an algorithm to calculate $f(1, n)$ in $O(n^3)$ time.

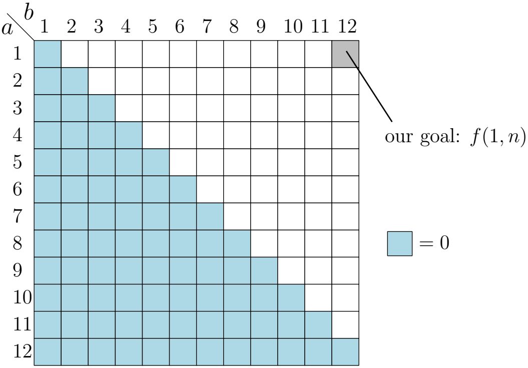
Solution

List all the subproblems.



Solution

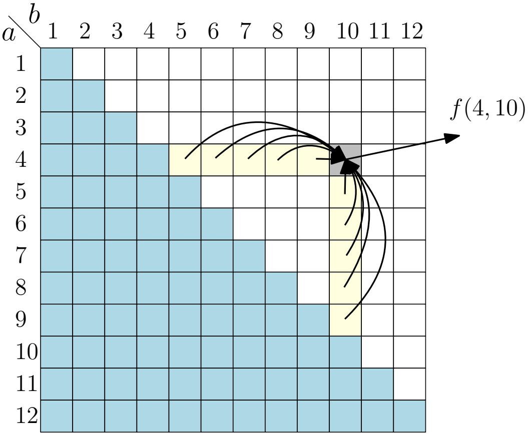
$f(a, b) = 0$ when $a \geq b$.



Solution

$f(a, b) = (\sum_{c=a}^{b} A[c]) + \min_{c=a+1}^{b-1}\{f(a, c) + f(c, b)\}$ when $a < b$.

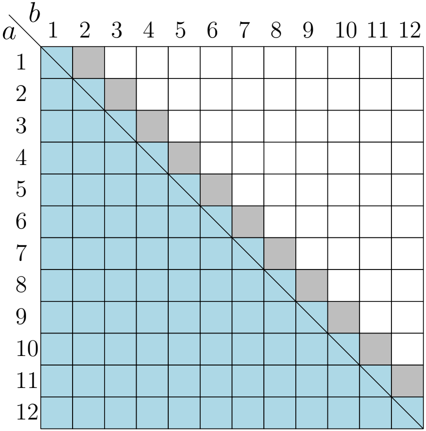
Find out the dependency relationships.



Solution

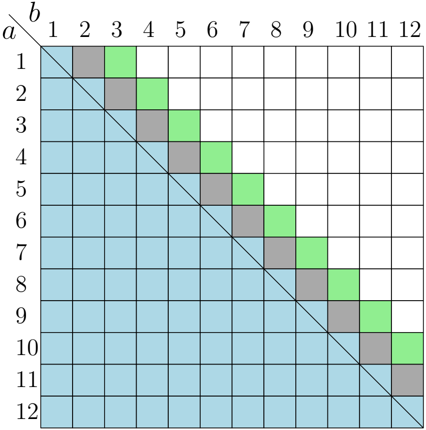
$f(a, b) = (\sum_{c=a}^{b} A[c]) + \min_{c=a+1}^{b-1}\{f(a, c) + f(c, b)\}$ when $a < b$.

Let us start with the gray cells — they correspond to $f(a, b)$ where $a = b - 1$. These cells depend on no other cells.



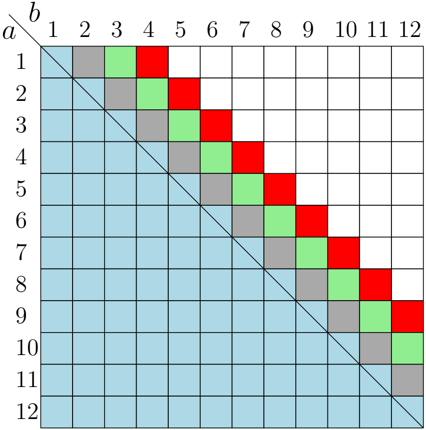
Solution

Let us continue with the green cells — they correspond to $f(a, b)$ where $a = b - 2$. Every such cell depends on two gray cells, which have already been computed.



Solution

Let us continue with the red cells — they correspond to $f(a, b)$ where $a = b - 3$. Every such cell depends on two gray cells and two green cells, all of which have been computed.



Solution

The order can be summarized as follows.

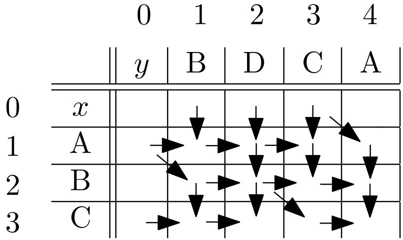
- All cells $f(a, b)$ with $b - a = 1$, each computed in $O(1)$ time.
- All cells $f(a, b)$ with $b - a = 2$, each computed in $O(2)$ time.
- ...
- All cells $f(a, b)$ with $b - a = k$, each computed in $O(k)$ time.
- ...
- All cells $f(a, b)$ with $b - a = n - 1$, each computed in $O(n - 1)$ time.

There are $O(n^2)$ values to calculate.
Total time complexity $= O(n^3)$.

Problem 4 (Space Consumption)

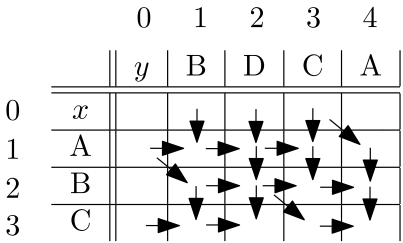
In Lecture Notes 8, our algorithm for computing $f(n, m)$ used $O(nm)$ space. Next, we will reduce the space complexity to $O(n + m)$.

Recall the dependency graph:



Solution

We can calculate the values in the row-major order, i.e., row 0 to row 3 and left to right in each row. We used $O(mn)$ space because we stored all the values. Observe, however, that only two rows need to be stored at any moment .



Solution

Same idea for the column-major order.

		0	1	2	3	4
		y	B	D	C	A
0	x					
1	A					
2	B					
3	C					

So the space complexity is $O(\min\{m, n\})$, in addition to the $O(n + m)$ space needed to store x and y.

Think: Can this trick be used to reduce the space in Problem 2?