CSCI3160: Regular Exercise Set 3

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Problem 1. Let S be a set of n intervals $\{[s_i, f_i] \mid 1 \leq i \leq n\}$, satisfying $f_1 \leq f_2 \leq ... \leq f_n$. Denote by S' the set of intervals in S that are disjoint with $[s_1, f_1]$. Prove: if $T' \subseteq S'$ is an optimal solution to the activity selection problem on S', then $T' \cup \{[s_1, f_1]\}$ is an optimal solution to the activity selection problem on S.

Problem 2. Describe how to implement the activity selection algorithm discussed in the lecture in $O(n \log n)$ time, where n is the number of input intervals.

Problem 3. Prof. Goofy proposes the following greedy algorithm to "solve" the activity selection problem. Let S be the input set of intervals. Initialize an empty T, and then repeat the following steps until S is empty:

- (Step 1) Add to T the interval $I = [s, f]$ in S that has the smallest s-value.
- (Step 2) Remove from S all the intervals overlapping with I (including I itself).

Finally, return T as the answer.

Prove: the above algorithm does not guarantee an optimal solution.

Problem 4.** Prof. Goofy just won't give up! This time he proposes a more sophisticated greedy algorithm. Again, let S be the input set of intervals. Initialize an empty T, and then repeat the following steps until S is empty:

- (Step 1) Add to T the interval $I \in S$ that overlaps with the *fewest* other intervals in S.
- (Step 2) Remove from S the interval I as well as all the intervals that overlap with I.

Finally, return T as the answer.

Prove: the above algorithm does not guarantee an optimal solution.

Problem 5* (Fractional Knapsack). Let (w_1, v_1), (w_2, v_2), ..., (w_n, v_n) be n pairs of positive real values. Given a real value $W \leq \sum_{i=1}^{n} w_i$, design an algorithm to find $x_1, x_2, ..., x_n$ to maximize the *objective function*

$$\sum_{i=1} \frac{x_i}{w_i} \cdot v_i$$

subject to

- $0 \leq x_i \leq w_i$ for every $i \in [1, n]$;
- $\sum_{i=1}^{n} x_i \leq W$.

Remark: You can imagine, for each $i \in [1, n]$ that the value w_i is the 'weight' of a certain item, and v_i is the item's 'value'. The goal is to maximize the total value of the items we collect, subject to the constraint that all the items must weight no more than W in total. For each item, we are allowed to take only a fraction of it, which reduces its weight and value by proportion.