The $k$-Selection Problem (Deterministic)

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The $k$-Selection Problem

### Input

You are given a set $S$ of $n$ integers in an array, the value of $n$, and also an integer $k \in [1, n]$.

### Output

The $k$-th smallest integer of $S$. 
We will describe an algorithm solving the problem deterministically in $O(n)$ time.

Recall:

Define the rank of an integer $v$ in $S$ as the number of elements in $S$ smaller than or equal to $v$.

For example, the rank of 23 in \{76, 5, 8, 95, 10, 31\} is 3, while that of 31 is 4.
A Deterministic Algorithm

We will assume that \( n \) is a multiple of 10 (if not, pad up to 9 dummy elements).

**Step 1:** Divide \( A \) into chunks of size 5, that is: (i) each chunk has 5 elements, and (ii) there are \( n/5 \) chunks.

**Step 2:** From each chunk, identify the median of the 5 elements therein. Collect all the \( n/5 \) medians into an array \( B \).

**Step 3:** Recursively run the algorithm to find the median \( p \) of \( B \).
A Deterministic Algorithm

**Step 4:** Find the rank $r$ of $p$ in $A$.

**Step 5:**

- If $r = k$, return $p$.
- If $r < k$, produce an array $A'$ containing all the elements of $A$ strictly less than $p$. Recursively find the $k$-th smallest element in $A'$.
- If $r > k$, produce an array $A'$ containing all the elements of $A$ strictly greater than $p$. Recursively find the ($k - r$)-th smallest element in $A'$. 

The $k$-Selection Problem (Deterministic)
Lemma 1.
The value of $r$ falls in the range from $\lceil (3/10)n \rceil$ to $\lceil (7/10)n \rceil + 7$.

**Proof:** Let us first prove the lemma by assuming that $n$ is a multiple of 10.

Let $C_1$ be the set of chunks whose medians are $\leq p$. Let $C_2$ be the set of chunks whose medians are $> p$.

Hence: $|C_1| = |C_2| = n/10$. 
Analysis

Every chunk in $C_1$ contains at least 3 elements $\leq p$. Hence:

$$r \geq 3|C_1| = (3/10)n.$$ 

Every chunk in $C_2$ contains at least 3 elements $> p$. Hence:

$$r \leq n - 3|C_1| = (7/10)n.$$ 

It thus follows that when $n$ is a multiple of 10, $r \in [(3/10)n, (7/10)n]$. 

The $k$-Selection Problem (Deterministic)
Now consider that $n$ is not a multiple of 10. Let $n'$ be the lowest multiple of 10 at least $n$. Hence, $n \leq n' < n + 10$. By our earlier analysis:

\[
(3/10)n' \leq r \leq (7/10)n'
\]

\[
\Rightarrow \quad (3/10)n \leq r \leq (7/10)(n + 10) = (7/10)n + 7
\]

\[
\Rightarrow \quad \lceil (3/10)n \rceil \leq r \leq (7/10)(n + 10) < \lceil (7/10)n \rceil + 7
\]

where the last step used the fact that $r$ is an integer.

\[\square\]
Analysis

Let $f(n)$ be the worst-case running time of our algorithm on $n$ elements.

We know that when $n$ is at most a certain constant, $f(n) = O(1)$.

For larger $n$:

$$f(n) = f(\lceil (n + 10)/5 \rceil) + f(\lceil (7/10)n \rceil + 7) + O(n)$$
$$= f(\lceil n/5 \rceil + 2) + f(\lceil (7/10)n \rceil + 7) + O(n)$$

Solving the recurrence gives $f(n) = O(n)$. 