Dynamic Programming 3: Edit Distances

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Remember that designing a dynamic programming algorithm requires discovering a recursive structure of the underlying problem. Today we will illustrate this through another problem: **computing the edit distance of two strings**.
Practical applications often need to evaluate the similarity of two strings. For example, when you mis-type “algorithm” as “algorhtm” at Google, you may be delighted that the search engine has corrected the spelling error for you. But why wouldn’t Google think that your mis-spelled word could be “structure”? The answer is, of course, “algorhtm” looks more similar to “algorithm” then to “structure”. To make such a clever judgement, we must resort to a metric to quantify string similarity.

We will discuss one popular metric: **edit distance**.
Edit Distance

Given two strings $s$ and $t$, the edit distance \( edit(s, t) \) is the smallest number of following edit operations to turn $s$ into $t$:

- **Insertion**: add a letter
- **Deletion**: remove a letter
- **Substitution**: replace a character with another one.
Example

Consider that \( s = \text{abode} \) and \( t = \text{blog} \). Then, \( \text{edit}(s, t) = 4 \) because

- We can change \text{abode} into \text{blog} by 4 operations:
  1. delete a \( \Rightarrow \) \text{bode}
  2. insert l after b \( \Rightarrow \) \text{blode}
  3. delete d \( \Rightarrow \) \text{bloe}.
  4. substitute e with g \( \Rightarrow \) \text{blog}

- Impossible to do so with at most 3 operations.

**Remark:** There could be more than one way to change \( s \) into \( t \) using the smallest number of operations. In the above example, try to come up with another 4 operations to change \text{abode} into \text{blog}. 
The Edit Distance Problem

**Input**: A string $s$ of $m$ letters, and a string $t$ of $n$ letters.

**Output**: Their edit distance $edit(s, t)$. 
Some Notations

To facilitate the subsequent discussion, let us agree on some notations.

Given a string $\sigma$, denote by

- $|\sigma|$ the length of $\sigma$, i.e., how many letters there are in $\sigma$.
- $\sigma[i]$ the $i$-th character of $\sigma$, for each $i \in [1, |\sigma|]$.
- $\sigma[x..y]$ as the substring of $\sigma$ starting from $\sigma[x]$ and ending at $\sigma[y]$. Specially, if $x > y$, then $\sigma[x..y]$ refers to the empty string.
**Lemma:** Let $s$ and $t$ be two strings with lengths $m$ and $n$, resp.

1. If $m = 0$, then $edit(s, t) = n$.
2. If $n = 0$, then $edit(s, t) = m$.
3. If $m > 0$, $n > 0$, and $s[m] = t[n]$, then $edit(s, t)$ is

   $$
   \min \left\{ 1 + edit(s, t[1..n-1]),
   1 + edit(s[1..m-1], t[1..n-1]),
   edit(s[1..m-1], t[1..n-1]) \right\}
   $$

4. If $m > 0$, $n > 0$, and $s[m] \neq t[n]$, then $edit(s, t)$ is

   $$
   \min \left\{ 1 + edit(s, t[1..n-1]),
   1 + edit(s[1..m-1], t[1..n-1]),
   1 + edit(s[1..m-1], t[1..n-1]) \right\}
   $$

We will prove the lemma at the end.
Calculating the recursive function in the preceding slide is a typical application of dynamic programming.
Before proceeding, let us observe several facts about the recurrence on Slide 8:

- Function $edit(.,.)$ has 2 parameters.

- The first parameter has $m + 1$ possible choices, namely, $s[1..0], s[1..1], \ldots, s[1..m]$.

- The second parameter has $n + 1$ possible choices, namely, $t[1..0], t[1..1], \ldots, t[1..n]$.

- In any case, $edit(a, b)$ depends only on $edit(a', b')$ where $a'$ and $b'$ are shorter than $a$ and $b$, respectively.

These observations motivate us to evaluate the recursion in a bottom-up manner: starting with the short strings and then propagating to the longer ones.
Dynamic Programming

Initialize a two-dimensional array $A$ of $m+1$ rows and $n+1$ columns. Label the rows as $0, \ldots, m$, and the columns as $0, \ldots, n$.

The algorithm aims to fill in the cell $A[i,j]$ at row $i$ and column $j$ as:

$$A[i, j] = edit(s[1..i], t[1..j]).$$

The value of $A[m, n]$ is therefore $edit(s, t)$. 
Example

The target matrix $A$ for $s = \text{abode}$ and $t = \text{blog}$:

$$
\begin{array}{cccccc}
& 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 2 & 3 & 4 \\
2 & 2 & 1 & 2 & 3 & 4 \\
3 & 3 & 2 & 2 & 2 & 3 \\
4 & 4 & 3 & 3 & 3 & 3 \\
5 & 5 & 4 & 4 & 4 & 4 \\
\end{array}
$$
Dynamic Programming

The algorithm fills in $A$ according to the order below:

1. Fill in row 0 and column 0.
2. Fill in the cells of row 1 from left to right.
3. Fill in the cells of row 2 from left to right.
4. ...
5. Fill in the cells of row $m$ from left to right.
The recurrence on Slide 8 guarantees that when we need to fill in a cell $A[i,j]$, all the dependent cells must have been ready.

Specifically, $A[i,j] =$

$$\min \begin{cases} 
1 + A[i, j - 1] \\
1 + A[i - 1, j] \\
A[i - 1, j - 1] \text{ if } s[i] = t[j], \text{ or } 1 + A[i - 1, j - 1] \text{ otherwise}
\end{cases}$$
Example

$s = \text{abode}$ and $t = \text{blog}$.
The matrix $A$ at the beginning:

\[
\begin{array}{c|cccc}
   & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & - & - & - & - & - \\
1 & - & - & - & - & - \\
2 & - & - & - & - & - \\
3 & - & - & - & - & - \\
4 & - & - & - & - & - \\
5 & - & - & - & - & - \\
\end{array}
\]
Example

$s = \text{abode}$ and $t = \text{blog}$.
Fill in column 0 and row 0:

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Example

$s = abode$ and $t = blog$. Now we fill in cell $A[1, 1]$. Since $s[1] = a$ which is different from $t[1] = b$, the recurrence on Lemma 8 says that $A[1, 1] =$

$$\min \left\{ \begin{array}{c}
1 + A[1, 0] = 1 \\
1 + A[0, 1] = 1 \\
1 + A[0, 0] = 1
\end{array} \right. = 1$$

which is 1.
Example

$s = \text{abode}$ and $t = \text{blog}$.
Similarly, fill in the other cells in row 1.

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Example

$s = \text{abode}$ and $t = \text{blog}$.
Now we fill in cell $A[2, 1]$. Since $s[1] = b$ which is the same as $t[1] = b$, the recurrence on Lemma 8 says that $A[2, 1] =$

$$
\min \left\{ \begin{array}{l}
1 + A[2, 0] = 3 \\
1 + A[1, 1] = 2 \\
A[1, 0] = 1
\end{array} \right. 
$$

which is 1.
**Example**

\[ s = \text{abode} \text{ and } t = \text{blog}. \]

Fill in the other cells of row 2.

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The algorithm then continues in the same fashion to fill in rows 3, 4, and 5.
Running Time

Clearly, filling in one cell takes only $O(1)$ time. As there are $O(nm)$ cells to fill, the overall running time is $O(nm)$. 
We now proceed to prove the lemma on Slide 8.
Proof: Cases 1 and 2 are trivial. We will focus on proving Case 3 because Case 4 can be established with a similar argument.

Henceforth, we will consider $m > 0$, $n > 0$, and $s[m] = t[n]$. 
We will first show

$$edit(s, t) \leq \min \left\{ 1 + edit(s, t[1..n-1]), 1 + edit(s[1..m-1], t), edit(s[1..m-1], t[1..n-1]) \right\}$$

In fact, this directly follows from the fact that we can convert $s$ into $t$ in 3 methods:

1. Delete $t[n]$, and use the least number of edit operations to change $s$ into $t[1..n-1]$. The total number of edit operations is therefore $1 + edit(s, t[1..n-1])$.

2. Delete $s[m]$, and use the least number of edit operations to change $s[1..m-1]$ into $t$. The total number of edit operations is therefore $1 + edit(s[1..m-1], t)$.

3. Simply change $s[1..m-1]$ into $t[1..n-1]$. The total number of edit operations is therefore $edit(s[1..m-1], t[1..n-1])$. 
The rest of the proof is to establish the following non-trivial fact:

\[
\text{edit}(s, t) \geq \min \left\{ \begin{array}{l}
1 + \text{edit}(s, t[1..n - 1]) \\
1 + \text{edit}(s[1..m - 1], t) \\
\text{edit}(s[1..m - 1], t[1..n - 1])
\end{array} \right. 
\]

which will complete the whole proof.
Let $SEQ^*$ be an optimal sequence of edit operations that converts $s$ into $t$. Denote by $|SEQ^*|$ the length of $SEQ^*$. Our objective is to prove that at least one of the following will happen:

1. We can obtain a sequence of $|SEQ^*| - 1$ edit operations that converts $s$ into $t[1..n - 1]$.

2. We can obtain a sequence of $|SEQ^*| - 1$ edit operations that converts $s[1..m - 1]$ into $t$.

3. We can obtain a sequence of $|SEQ^*|$ edit operations that converts $s[1..m - 1]$ into $t[1..n - 1]$.

This will establish the inequality of the previous slide (think: why?).
We will distinguish three possibilities.

**Possibility 1:** $SEQ^*$ never deletes or replaces $s[m]$.  
In this case, $SEQ^*$ itself is a sequence of operations that converts $s[1..m-1]$ into $t[1..n-1]$; hence, Case 3 happens.
**Possibility 2: \( SEQ^* \) deletes \( s[m] \).**

In this case, after discarding the operation deleting \( s[m] \), \( SEQ^* \) becomes a sequence of operations that converts \( s[1..m-1] \) into \( t \); hence, Case 2 happens.
Possibility 3: $SEQ^*$ replaces $s[m]$ with a character say $\Delta \neq s[m]$.

**Claim 1:** $\Delta$ is then never deleted or replaced in $SEQ^*$.

**Proof:** If $\Delta$ is deleted later, then we can make $SEQ^*$ shorter by directly removing $s[m]$ with one single operation (thus saving two operations: replacing $s[m]$ with $\Delta$ and then deleting $\Delta$).

If $\Delta$ is replaced with $\Delta'$ later, then we can make $SEQ^*$ shorter by directly replacing $s[m]$ with $\Delta'$ (thus saving two operations: replacing $s[m]$ with $\Delta$ and then with $\Delta'$).

**Think:** Why can we assert $\Delta \neq s[m]$?
Claim 2: \( SEQ^* \) must contain an operation “insert \( t[n] \)” that inserts the character matching \( t[n] \) at the end.

**Proof:** By Claim 1, \( \Delta \) remains in the final string obtained by \( SEQ^* \). As \( \Delta \neq s[m] \), the final string must contain a character — say \( c \) — that matches \( t[n] \). Since no operations can change the order of two characters, that character \( c \) must have been inserted by \( SEQ^* \).

The character \( c \) must be inserted by the operation “insert \( t[n] \)”.

Otherwise, suppose that \( c \) was inserted by “insert \( \Delta'' \)” for some \( \Delta'' \neq t[n] \). There would have to be another operation later in \( SEQ^* \) that replaced \( c \) with \( t[n] \). We could then make \( SEQ^* \) shorter by replacing the first operation with “insert \( t[n] \)” and removing the latter one.
In this case, after discarding the operation described in Claim 2, $SEQ^*$ becomes a sequence of operations that converts $s$ into $t[1..n-1]$; hence, Case 1 happens.