CSCI3160: Regular Exercise Set 2

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Problem 1 (Faster Algorithm for Finding the Number of Crossing Inversions). Let $S_1$ and $S_2$ be two disjoint sets of $n$ integers. Assume that $S_1$ is stored in an array $A_1$, and $S_2$ in an array $A_2$. Both $A_1$ and $A_2$ are sorted in ascending order. Design an algorithm to find the number of such pairs $(a, b)$ satisfying all of the following conditions: (i) $a \in S_1$, (ii) $b \in S_2$, and (iii) $a > b$. Your algorithm must finish in $O(n)$ time (we gave an $O(n \log n)$-time algorithm in the class).

Solution. Merge $A_1$ and $A_2$ into one sorted list $A$, which takes $O(n)$ time. Scan the elements of $A$ in ascending order. In the meantime, maintain the number $n_2$ of elements that (i) originate from $A_2$, and (ii) have already been scanned so far: this can be done by setting $n_2$ to 0 at the beginning, and increment it each time an element originating from $A_2$ is scanned. Furthermore, also maintain a counter $c$ as follows: $c = 0$ at the beginning; every time an element $a$ originating from $A_1$ is seen, $c$ is increased by the current value of $n_2$. The final $c$ at the end of the algorithm is the answer we are looking for.

Problem 2. Give an $O(n \log n)$-time algorithm to solve the dominance counting problem discussed in the class. (Hint: Using the result of Problem 1)

Solution. Let $P$ be the input set of points. Recall that, as discussed in the class, our algorithm divides $P$ into two halves $P_1$ and $P_2$ using a vertical line $\ell$, and then recurse on $P_1$ and $P_2$ respectively. The first change we make to the algorithm is to ensure that, when the recursion on $P_1$ and $P_2$ ends, the points of $P_1$ and $P_2$ have been sorted by y-coordinate. Now it remains to find, for each point $p_2 \in P_2$, the number of points $p_1 \in P_1$ that are dominated by $p_2$. Next we show that this can be done in $O(n)$ time, which makes the total running time $O(n \log n)$.

In $O(n)$ time, merge $P_1$ and $P_2$ into one sorted list $P$, where the points are sorted in ascending order by y-coordinate. Scan $P$. In the meantime, maintain the number $n_1$ of points that (i) originate from $P_1$, and (ii) have already been scanned so far. Every time a point $p_2$ originating from $P_2$ is seen, the number of points $p_1 \in P_1$ dominated by $p_2$ is precisely the current value of $n_1$.


1. Give an algorithm to find a sub-array of with the largest weight, among all sub-arrays $A[i : j]$ with $j = n$. Your algorithm must finish in $O(n)$ time.

2. Give an algorithm to find a sub-array with the largest weight in $O(n \log n)$ time (among all the possible sub-arrays).

Solution. Subproblem 1: Scan the elements of $A$ from $A[n]$ to $A[1]$. At any time, maintain the sum $s$ of the elements already scanned: at the beginning $s = 0$; after scanning an element $A[i]$, add $A[i]$ to $s$. Every time we finish doing so for element $A[i]$, the current value $s$ is precisely the weight
of $A[i : n]$. In this way, we obtain the weights of all sub-arrays $A[n : n]$, $A[n - 1 : n]$, ..., $A[1 : n]$ (in this order) in $O(n)$ time. The maximum weight can then be found easily.

Subproblem 2: Break $A$ into two halves: array $A_1$ which contains the first $\lceil n/2 \rceil$ elements, and array $A_2$ which contains the rest. Recursively, find the sub-array of $A_1$ with the largest weight, and then the sub-array of $A_2$ with the largest weight. It remains to consider the “crossing” sub-arrays $A[i : j]$ where $i \leq \lceil n/2 \rceil$ and $j > \lceil n/2 \rceil$. In particular, we want to find the “best” crossing sub-array, i.e., the one with the maximum weight. Then, the sub-array to output can be decided easily from the three sub-arrays aforementioned.

We say that a sub-array $A_1[i : j]$ is right grounded if $j = \lceil n/2 \rceil$, and a sub-array $A_2[i : j]$ is left grounded if $i = 1$. A crucial observation is that the “best” crossing sub-array must be the concatenation of

- the right grounded sub-array in $A_1$ with the maximum weight, and
- the left grounded sub-array in $A_2$ with the maximum weight.

From Subproblem 1, we know that each of the above two grounded sub-arrays can be found in $O(n)$ time.

Therefore, if $f(n)$ is the time of solving the problem on an array of length $n$, it holds that $f(n) \leq 2 \cdot f(\lceil n/2 \rceil) + O(n)$, which gives $f(n) = O(n \log n)$.

**Problem 4.** In the class, we explained how to multiply two $n \times n$ matrices in $O(n^{2.81})$ time when $n$ is a power of 2. Explain how to ensure the running time for any value of $n$.

**Solution.** If $n$ is not a power of 2, let $m$ be the smallest power of 2 that is larger than $n$. If $A, B$ are the $n \times n$ input matrices, obtain an $m \times m$ matrix $A'$ by padding $m - n$ dummy rows and columns to $A$ containing only 0 values, and similarly, an $m \times m$ matrix $B'$ from $B$. Calculate $A'B'$ in $O(m^{2.81}) = O((2n)^{2.81}) = O(n^{2.81})$ time. Then, the matrix $AB$ can be obtained by discarding the last $m - n$ rows and columns from the matrix $A'B'$. 
