Week 3 Tutorial

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The Predecessor Search Problem

Problem Input

- An array A of n integers in ascending order
- A search value q

Goal:

Find the predecessor of q in A.

Remark: the predecessor of q is the largest element in A that is smaller than or equal to q.

Example

1. If $q = 23$, the predecessor is 21.
2. If $q = 21$, the predecessor is also 21.
3. If $q = 1$, no predecessor.

2	3	5	8	13	21	34	55

A

Binary Search

- If A contains q, binary search will find q directly.
- If A does not contain q, the predecessor of q can be easily inferred from where the algorithm terminates.

2	3	5	8	13	21	34	55

A

The Two-Sum Problem

Input

- An array of n integers in ascending order.
- An integer v.

Goal:

Determine whether A contains two different integers x and y such that $x + y = v$.

Example

- If $v = 30$, answer "yes".
- If $v = 29$, answer "no".

2	3	5	7	11	13	17	19	23	29	31	37

Solution

Use binary search as a building brick.

Key idea: For each x in the array, look for $v - x$ with binary search.

Analysis

This algorithm performs at most n binary searches.

Cost of the algorithm: $O(n \log n)$

Can you do even better?
Try to solve this problem in $O(n)$ time (not covered in this tutorial).

More on big-O

Recall the definition of $f(n) = O(g(n))$:

> $f(n) = O(g(n))$, if there exist two positive constants c_1 and c_2 such that $f(n) \leq c_1 \cdot g(n)$ holds for all $n \geq c_2$.

Another approach is to compute $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ and decide as follows:

- $f(n) = O(g(n))$, if the limit is bounded by an constant;
- $f(n) \neq O(g(n))$, if the limit is ∞.

Note: there is a third possibility for the limit, where the approach will fail.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$.

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Method 1: Constant finding

1. Fix c_1
2. Solve for c_2
3. If a c_2 cannot be found, go back to Step 1 and try a different c_1.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$

(try $c_1 = 5$)

$$
\begin{aligned}
& & f(n) \leq c_1 \cdot g(n) \\
\Leftrightarrow & & 10n + 5 \leq c_1 \cdot n^2 \\
\Leftrightarrow & & 5(2n + 1) \leq 5 \cdot n^2 \\
\Leftrightarrow & & 2n + 1 \leq n^2 \\
\Leftrightarrow & & 2 \leq (n - 1)^2 \\
\Leftarrow & & 3 \leq n
\end{aligned}
$$

Hence, it suffices to set $c_2 = 3$.

Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$.

Method 2: Limit

$$\lim_{n \to \infty} \frac{10n + 5}{n^2} = \lim_{n \to \infty} \frac{10 + 5/n}{n} = 0.$$

Hence, $f(n) = O(g(n))$.

Exercise 2

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $g(n) \neq O(f(n))$.

Method 1: Constant finding (prove by contradiction)

Suppose that $g(n) = O(f(n))$, i.e., there are constants c_1, c_2 such that, for all $n \geq c_2$, we have

$$\begin{aligned} & n^2 \leq c_1 \cdot (10n + 5) \\ \Rightarrow \quad & n^2 \leq c_1 \cdot 20n \\ \Leftrightarrow \quad & n \leq 20c_1 \end{aligned}$$

which cannot hold for all $n \geq c_2$, regardless of c_2. This gives a contradiction.

Exercise 2

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $g(n) \neq O(f(n))$.

Method 2: Limit

$$\lim_{n \to \infty} \frac{n^2}{10n + 5} = \infty.$$

Hence, $g(n) \neq O(f(n))$.

In some rare scenarios, the limit approach may fail. We will see an example next.

Consider $f(n) = 2^n$. Define $g(n)$ as:

- $g(n) = 2^n$ if n is even;
- $g(n) = 2^{n-1}$ otherwise.

Since $f(n) \leq 2g(n)$ holds for all $n \geq 1$, it holds that $f(n) = O(g(n))$.

However, $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ does not exist, because it keeps jumping between 1 and 2 as n increases!

Next, we discuss how to extend the big-O definition to two variables. The definition can be extended to more variables following the same idea.

Big-O with Two Variables

Let $f(n, m)$ and $g(n, m)$ be functions of variables n and m satisfying $f(n, m) \geq 0$ and $g(n, m) \geq 0$. We say $f(n, m) = O(g(n, m))$ if there exist constants c_1 and c_2 such that $f(n, m) \leq c_1 \cdot g(n, m)$ holds for all $n \geq c_2$ and $m \geq c_2$.

Regular Excercise 2 Problem 8

Let $f(n, m) = n^2m + 100nm$ and $g(n, m) = n^2m$.
Prove $f(n, m) = O(g(n, m))$.

Obviously:

$$n^2m + 100nm \quad \leq \quad 101n^2m$$

for any $n \geq 1$ and $m \geq 1$.
Hence, it suffices to set $c_1 = 101$ and $c_2 = 1$.

Let $f(n, m) = n^2 m + 100nm^2$ and $g(n, m) = n^2 m + nm^2$.
Prove $f(n, m) = O(g(n, m))$.

Obviously:

$$n^2 m + 100nm^2 \quad \leq \quad 100(n^2 m + nm^2)$$

for any $n \geq 1$ and $m \geq 1$.
Hence, it suffices to set $c_1 = 100$ and $c_2 = 1$.