Implementation of Dijkstra's Algorithm

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In the lecture, we have proved that Dijkstra's algorithm computes the shortest distances correctly. Today, we will see how the algorithm can be slightly augmented to output a shortest path tree. Furthermore, we will also discuss how to implement the algorithm in $O((|V| + |E|) \log |V|)$ time.

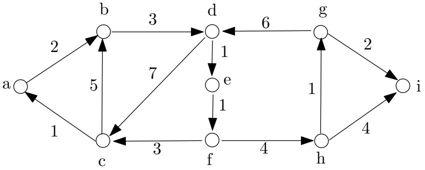
Single Source Shortest Path (SSSP) with Positive Weights

Let (G, w) with $G = (V, E)$ be a directed weighted graph, where w maps every edge of E to a positive value.

Given a vertex s in V, the goal of the **SSSP problem** is to find, for every other vertex $t \in V \setminus \{s\}$, a shortest path from s to t, unless t is unreachable from s.

Example

Suppose that the source vertex is a.

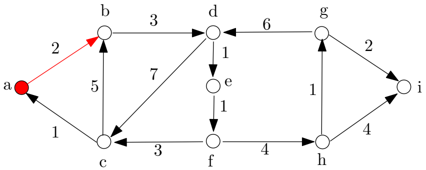


vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	∞	nil
c	∞	nil
d	∞	nil
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$F = \emptyset$ and
$S = \{a, b, c, d, e, f, g, h, i\}$.
Since $dist(a)$ is the smallest among those of vertices in P, pick a.

Example

Relax the out-going edges of a. **Red** edges correspond to the parent column.

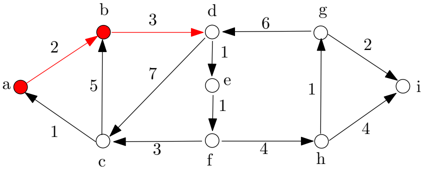


$F = \{a\}$ (vertices finalized) and $S = \{b, c, d, e, f, g, h, i\}$.

vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	$\infty \to 2$	nil $\to a$
c	∞	nil
d	∞	nil
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

Example

Relax the out-going edges of b. **Red** edges correspond to the parent column.

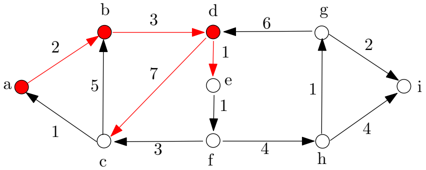


$F = \{a, b\}$ and
$S = \{c, d, e, f, g, h, i\}$.

vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	∞	nil
d	$\infty \to 5$	nil $\to b$
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

Example

Relax the out-going edges of d. **Red** edges correspond to the parent column.

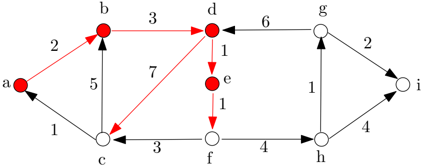


$F = \{a, b, d\}$ and
$S = \{c, e, f, g, h, i\}$.

vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	$\infty \rightarrow 12$	nil $\rightarrow d$
d	5	b
e	$\infty \rightarrow 6$	nil $\rightarrow d$
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

Example

Relax the out-going edges of e. **Red** edges correspond to the parent column.

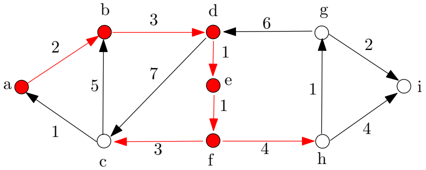


$F = \{a, b, d, e\}$ and
$S = \{c, f, g, h, i\}$.

vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	12	d
d	5	b
e	6	d
f	$\infty \to 7$	nil $\to e$
g	∞	nil
h	∞	nil
i	∞	nil

Example

Relax the out-going edges of f. **Red** edges correspond to the parent column.

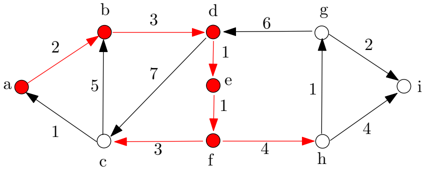


$F = \{a, b, d, e, f\}$ and
$S = \{c, g, h, i\}$.

vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	$12 \rightarrow 10$	$d \rightarrow f$
d	5	b
e	6	d
f	7	e
g	∞	nil
h	$\infty \rightarrow 11$	nil $\rightarrow f$
i	∞	nil

Example

Relax the out-going edges of c. **Red** edges correspond to the parent column.

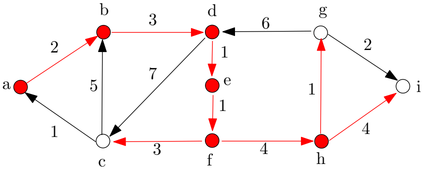


$F = \{a, b, c, d, e, f\}$ and $S = \{g, h, i\}$.

vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	∞	nil
h	11	f
i	∞	nil

Example

Relax the out-going edges of h. **Red** edges correspond to the parent column.

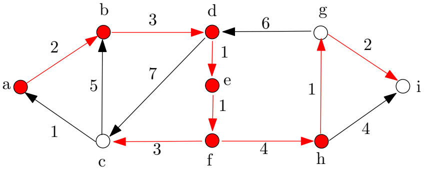


$F = \{a, b, c, d, e, f, h\}$ and $S = \{g, i\}$.

vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	$\infty \to 12$	nil $\to h$
h	11	f
i	$\infty \to 15$	nil $\to h$

Example

Relax the out-going edges of g. **Red** edges correspond to the parent column.

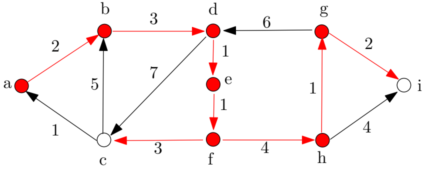


$F = \{a, b, c, d, e, f, g, h\}$ and $S = \{i\}$.

vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	12	h
h	11	f
i	$15 \rightarrow 14$	$h \rightarrow g$

Example

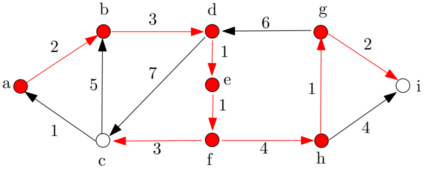
Relax the out-going edges of i. **Red** edges correspond to the parent column.



vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	12	h
h	11	f
i	14	g

$F = \{a, b, c, d, e, f, g, h, i\}$
and
$S = \{\}$.
Done.

Example



vertex v	$dist(v)$	$parent(v)$
a	0	nil
b	2	a
c	10	f
d	5	b
e	6	d
f	7	e
g	12	h
h	11	f
i	14	g

The set of red edges indicates the shortest path tree that we should output. **Think:** Given the table's last column, how can you find this tree in $O(|V|)$ time?

Next, we will discuss how to implement the algorithm in $O((|V| + |E|) \log |V|)$ time. For this purpose, we will assume that each vertex in V carries an ID from 1 to $n = |V|$.

Data Structures

Recall that S is the set of vertices that have not been finalized.

We create an array A of size n where, for each $v \in [1, n]$, the cell $A[v]$ stores $dist(v)$ and $parent(v)$.

Define $D = \{dist(v) \mid v \in S\}$. We store D in an AVL-tree T.

All these data structures can be constructed at the beginning of Dijkstra's algorithm in $O(n \log n)$ time.

To run Dijkstra's, we need to support two operations:

- **DecreaseKey**(v, k_{new}): reduce $dist(v)$ to k_{new} in D if $k_{new} < dist(v)$.
- **DeleteMin**: remove the smallest $dist(v)$ from D.

We perform DecreaseKey at most $|E|$ times and DeleteMin at most $|V|$ times.

Think: Why?

Implementation of DecreaseKey

Goal: Reduce $dist(v)$ to k_{new} in D if $k_{new} < dist(v)$.

1. Find the current $dist(v)$ from array A
2. If $dist(v) \leq k_{new}$, do nothing and return
3. Delete $dist(v)$ from the AVL-tree T
4. Set $A[v] = k_{new}$
5. Insert k_{new} into T

$O(\log |V|)$ time.

Implementation of DeleteMin

Goal: Remove the smallest $dist(v)$ from D

Finding the smallest key in an AVL-tree takes only $O(\log |V|)$ time. After that, we can remove the key.

Two Minor Issues

- What if D has duplicate $dist(v)$ values?
- In DeleteMin, we also need to identify the vertex v whose $dist(v)$ is the smallest in D.

How to solve these issues (easily)?