# CSCI2100: Regular Exercise Set 11

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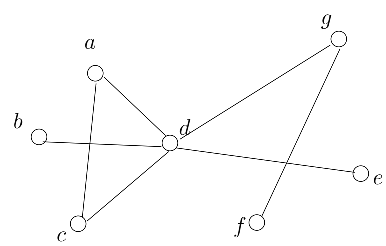
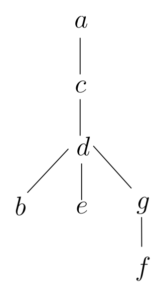
**Problem 1 (DFS on Undirected Graphs).** Let $G = (V, E)$ be an undirected graph. Consider the execution of DFS on $G$. The algorithm runs in exactly the same way as DFS on a directed graph. The only difference is that, a vertex $u$ is popped out of the stack, only if none of its neighbors (instead of out-neighbors) is still white. Give a possible DFS tree produced if we (i) start DFS on $a$ in the following graph, and (ii) follow the convention that we explore the neighbors of a vertex in alphabetic order.




**Solution.**




**Problem 2 (No Cross Edges in Undirected DFS).** Let $G = (V, E)$ be an undirected graph. Consider the DFS forest produced by running DFS on $G$ (assuming arbitrary starting and re-starting vertices). Let $\{u, v\}$ be an edge in $G$ (note that we use the notation $\{u, v\}$, instead of $(u, v)$, to emphasize that the edge has no directions). Prove: either $u$ is an ancestor of $v$, or $v$ is an ancestor of $u$.

*Remark:* Because of this lemma, we can classify each edge $\{u, v\}$ in $G$ as follows:

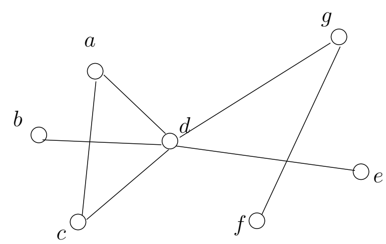
- *Tree edge*: if $u$ is the parent of $v$ or $v$ is the parent of $u$.
- *Back edge*: otherwise.

**Solution.** The white path theorem—as stated in Problem 1—still holds for undirected DFS (the same proof applies here as well). Between $u$ and $v$, let $u$ be the vertex discovered first. Then, the white path theorem says that $v$ must be a descendant of $u$.

**Problem 3 (Undirected Cycle Detection).** Let $G = (V, E)$ be an undirected graph. A *cycle* is a sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{t-1}, v_t\}$ where $v_t = v_1$. Adapt DFS to design an algorithm to detect whether $G$ has a cycle in $O(|V| + |E|)$ time.

**Solution.** Perform DFS on $G$. Declare cycle presence if and only if a back edge is found. For example, in the Solution of Problem 2, there is such an edge $\{a, d\}$, which implies a cycle.

**Problem 4** (Articulation Vertex).** Let $G = (V, E)$ be an undirected graph that is connected (i.e., there is a path between any two distinct vertices). A vertex $u \in V$ is called an *articulation vertex* if the following is true: $G$ becomes disconnected after removing $u$ and all the edges of $u$. For example, in the figure below, vertex $g$ is an articulation, and so is $d$. No other vertices are articulation vertices.




Consider any DFS tree on $G$. Prove:

- If a vertex $u$ is a leaf in the DFS tree, it cannot be an articulation vertex.
- Let $u$ a vertex that is neither a leaf in the DFS tree nor the root. It is an articulation vertex if and only if the following is true:
  - There is at least one child $v$ of $u$, such that no back edge connects a descendant of $v$ to a proper ancestor of $u$.
- Let $u$ be the root of a DFS tree. It is an articulation vertex if and only if it has at least two child nodes in the DFS tree.

**Solution.**
*Proof of the First Bullet.*
Suppose that $u$ is an articulation vertex. Let $s$ be the starting vertex of the DFS. Then there must be a vertex $u'$ such that all the paths from $s$ to $u'$ must go by way of $u$. This implies that, when $v$ is discovered by DFS, there must be a white path from $u$ to $u'$. The white path theorem then says that $u'$ must be a descendant of $u$, contradicting the fact that $u$ is a leaf.

*Proof of the Second Bullet.*
*Only-if direction.* Imagine removing $u$ from $G$, which should disconnect $G$. Let $C_1, C_2, ..., C_t$ for some $t \geq 2$ be the connected components (CCs) of the resulting graph (recall that a CC is a set of vertices that are reachable from each other). Without loss of generality, assume that $s$ belongs to $C_1$. Consider the moment right before the first vertex $v$ in $C_2$ is discovered. It must be a child of $u$ in the DFS tree (because any path from $s$ to $u$ must cross the edge $\{u, v\}$). At this moment, all the vertices in $C_2$ must be white; and they are the only vertices that $v$ can reach via white paths. Hence, all the vertices of $C_2$ must be the *only* descendants of $v$. It thus follows that there can be no back edge connecting a descendant of $v$ to a proper ancestor of $u$.

*If direction.* We will prove that, after $u$ is removed from $G$, $s$ can no longer reach $v$, which thus indicates that $u$ is an articulation vertex. Suppose, on the contrary, that $u$ can still access $v$ by a path $\pi$ (that does not contain $u$). Denote the vertices on $\pi$ as $v_1, v_2, ..., v_x$ with $v_1 = s$ and $v_x = v$.

Let $v_i$ (for some $i \in [1, x]$) be the *last* vertex on $\pi$ that is an ancestor of $u$. We will prove that $v_{i+1}$ must be a descendant of $v$, making $\{v_i, v_{i+1}\}$ a back edge that connects a descendant of $v$ to a proper ancestor of $u$, which contradicts the fact that no such back edges exist.

Consider the moment right before the discovery of $v$. We argue that the colors of $v_{i+1}, v_{i+2}, ..., v_{x-1}$ must all be white at this moment:

- First, none of them can be gray—otherwise, such a vertex must be an ancestor of $u$ (because $u$ is the parent of $v$), contradicting the definition of $v_i$.
- If $v_{i+1}$ is black, it means that $v_{i+1}$ was discovered before $v$. Furthermore, when $v_{i+1}$ turned black, $v_{i+2}$ cannot be white (in non-directed DFS, a vertex can turn black only if it has no white neighbors). Thus, at the moment when $v$ is discovered, $v_{i+2}$ must be black (as mentioned, $v_{i+2}$ cannot be gray). Following the same argument, we obtain that $v_{i+3}, v_{i+4}, ..., v_x$ must all be black at the moment. However, this contradicts the fact that $v_x = v$ is white.
- The same argument proves that none of $v_{i+2}, v_{i+3}, ..., v_{x-1}$ can be black.

Therefore, all of $v_{i+1}, v_{i+2}, ..., v_{x-1}$ must be descendants of $v$.

*Proof of the Third Bullet.*
*Only-if direction.* Vertex $u$ is the starting vertex of DFS. Imagine removing $u$ from $G$, which should disconnect $G$ into CCs $C_1, C_2, ..., C_t$ for some $t \geq 2$. Let $v$ be the second vertex discovered by DFS (i.e., right after $u$). Without loss of generality, suppose that $v \in C_1$. Then, when $v$ is discovered, there is no white path from $v$ to any vertex in $C_2$. Hence, none of the vertices in $C_2$ can be descendants of $v$, implying that $u$ must have another child.

*If direction.* Let $v$ be the second vertex discovered by DFS (i.e., right after $u$). Let $v'$ any other child of $u$ in the DFS tree. We will prove that any path from $v$ to $v'$ must go through $u$, which indicates that $u$ is an articulation vertex.

Assume that there is a path $\pi$ from $v$ to $v'$ that does not go through $u$. Then, when $v$ is discovered, there is a white path from $v$ to $v'$, which means that $v'$ must be a descendant of $v$ in the DFS tree. This contradicts the fact that $v'$ and $v$ are siblings.

**Problem 5* (Finding an Articulation Vertex).** Let $G = (V, E)$ be an undirected graph that is connected. Design an algorithm to determine whether $G$ has any articulation vertex. Your algorithm must finish in $O(|V| + |E|)$ time.

**Solution.** First grow a DFS-tree $T$, but make sure that at each node $u$ we record its level (the root is at level 0), denoted as *level*($u$). We now process the vertices of $T$ in a bottom-up manner (i.e., descending order of level). Let $u$ be a vertex to be processed next. We do the following:

- *Case 1: $u$ is a leaf node*: We inspect all the edges $\{u, v\}$ of $u$, and obtain:

$$\textit{highest-back-level}(u) = \min_{\text{all } \{u,v\}} \textit{level}(v).$$

- *Case 2: $u$ is an internal node but not the root*: Let $v_1, v_2, ..., v_t$ be its children (which have already been processed). If

$$\max_{i=1}^{t} \textit{highest-back-level}(v_i) \geq \textit{level}(u)$$

we report $u$ as an articulation vertex, and finish.

Otherwise, inspect all the edges $\{u, v\}$ of $u$, and obtain:

$$\textit{highest-back-level}(u) = \min_{\text{all } \{u,v\}} \textit{level}(v).$$

Then, update *highest-back-level*$(u)$ to be:

$$\min \left\{ \textit{highest-back-level}(u), \min_{i=1}^{t} \textit{highest-back-level}(v_i) \right\}.$$

- *Case 3: u is the root*: Report $u$ as an articulation vertex if it has at least 2 child nodes.