The $k$-Selection Problem

**Problem:** You are given a set $S$ of $n$ integers in an array and also an integer $k \in [1, n]$. Design an algorithm to find the $k$-th smallest integer of $S$.

For example, suppose that $S = (53, 92, 85, 23, 35, 12, 68, 74)$ and $k = 3$. You should output 35.

This problem can be easily settled in $O(n \log n)$ time by sorting. Next, we will solve it in $O(n)$ expected time with randomization.
Idea

To illustrate the idea behind our algorithm, suppose that we pick an arbitrary element (say the first) \( v \) of \( S \).

Move elements around so that those smaller than \( v \) are placed before \( v \), and those larger are placed after \( v \). This requires only \( O(n) \) time (no sorting required).

- If \( x = k - 1 \), done (\( v \) is what we are looking for).
- If \( x < k - 1 \), recurse by performing \((k - (x + 1))\)-selection on the \( y \) elements to the right of \( v \) (subproblem).
- If \( x > k - 1 \), recurse by performing \( k \)-selection on the \( x \) elements to the left of \( v \) (subproblem).
Idea

Obstacle: $x$ or $y$ can be very small (0 if we are unlucky) such that we can throw away only few elements before recursion.

Wish: Make $x \geq n/3$ and $y \geq n/3$.
Antidote: Randomly select $v$ from the whole array! Wish comes true with probability $1/3$!

New obstacle: Would still fail with probability $2/3$.
New antidote: Choose another $v$ if we fail; 3 repeats in expectation!
The **rank** of an integer $v$ in $S$ is the number of elements in $S$ smaller than or equal to $v$.

For example, suppose that $S = (53, 92, 85, 23, 35, 12, 68, 74)$. Then, the rank of 53 is 4 and that of 12 is 1.

Finding the rank of $v$ in $S$ takes only $O(|S|)$ time.
Algorithm

1. Randomly pick an integer \( v \) from \( S \); call \( v \) the pivot.
2. Get the rank \( r \) of \( v \).
3. If \( r \) is not in \([n/3, 2n/3]\), repeat from Step 1.
4. Otherwise:
   4.1 If \( k = r \), return \( v \).
   4.2 If \( k < r \), perform \( k \)-selection on the elements of \( S \) less than \( v \).
   4.3 If \( k > r \), perform \((k - r)\)-selection on the elements of \( S \) greater than \( v \).
Example

Goal: find the 10-th smallest element from 12 elements:

\[
\begin{array}{cccccccccc}
17 & 26 & 38 & 28 & 41 & 72 & 83 & 88 & 5 & 9 \\
\end{array}
\]

Suppose that the pivot \( v \) chosen happens to be 12, whose rank is 3, outside the range \([4, 8]\). We repeat by randomly choosing another pivot \( v \), which — let us assume — happens to be 83. Again, its rank 11 is outside the range \([4, 8]\). Repeat another time; let the pivot \( v \) returned by 35, whose rank is 7.

We recurse by finding the 3-rd smallest element in:

\[
\begin{array}{cccc}
38 & 41 & 72 & 83 \\
\end{array}
\]
Cost Analysis

Step 1 (on Slide 6) takes $O(1)$ time.
Step 2 takes $O(n)$ time.

How many times do we have to repeat the above two steps?
With a probability $1/3$, we can proceed to Step 3 $\Rightarrow$ need to repeat only 3 times in expectation!

When we are at Step 3, $A$ has at most $\lceil 2n/3 \rceil$ elements left.
Let $f(n)$ be the expected running time of our algorithm on an array of size $n$.

We know from the earlier analysis:

$$f(1) \leq O(1)$$
$$f(n) \leq O(n) + f(\lceil 2n/3 \rceil).$$

Solving the recurrence gives $f(n) = O(n)$ (Master’s theorem).
It is worth mentioning that the k-selection problem can be solved in $O(n)$ time deterministically. However, the algorithm is much more complicated. This demonstrates the power of randomization again.