CSCI2100: Regular Exercise Set 3

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**Problem 1.** Prove $\log_2(n!) = \Theta(n \log n)$.

**Problem 2.** Let $f(n)$ be a function of positive integer $n$. We know:

\[
\begin{align*}
  f(1) &= 1 \\
  f(n) &\leq 2 + f(\lceil n/10 \rceil).
\end{align*}
\]

Prove $f(n) = O(\log n)$. Recall that $\lceil x \rceil$ is the ceiling operator that returns the smallest integer at least $x$.

**Problem 3.** Let $f(n)$ be a function of positive integer $n$. We know:

\[
\begin{align*}
  f(1) &= 1 \\
  f(n) &\leq 2 + f(\lceil 3n/10 \rceil).
\end{align*}
\]

Prove $f(n) = O(\log n)$. Recall that $\lceil x \rceil$ is the ceiling operator that returns the smallest integer at least $x$.

**Problem 4.** Let $f(n)$ be a function of positive integer $n$. We know:

\[
\begin{align*}
  f(1) &= 1 \\
  f(n) &\leq 2n + 4f(\lceil n/4 \rceil).
\end{align*}
\]

Prove $f(n) = O(n \log n)$.

**Problem 5 (Bubble Sort).** Let us re-visit the sorting problem. Recall that, in this problem, we are given an array $A$ of $n$ integers, and need to re-arrange them in ascending order. Consider the following bubble sort algorithm:

1. If $n = 1$, nothing to sort; return.


Prove that the algorithm terminates in $O(n^2)$ time.

As an example, support that $A$ contains the sequence of integers (10, 15, 8, 29, 13). After Step 2 has been executed once, array $A$ becomes (10, 8, 15, 13, 29).

**Problem 6* (Modified Merge Sort).** Let us consider a variant of the merge sort algorithm for sorting an array $A$ of $n$ elements (we will use the notation $A[i..j]$ to represent the part of the array from $A[i]$ to $A[j]$):

- If $n = 1$ then return immediately.
- Otherwise set $k = \lceil n/3 \rceil$.
- Recursively sort $A[1..k]$ and $A[k+1..n]$, respectively.
- Merge $A[1..k]$ and $A[k+1..n]$ into one sorted array.

Prove that this algorithm runs in $O(n \log n)$ time.