More on Binary Heaps

CSCI2100 Tutorial 9

Jianwen Zhao

Department of Computer Science and Engineering
The Chinese University of Hong Kong

Adapted from the slides of the previous offerings of the course
In the previous lectures, we have implemented the priority queue (which supports \texttt{insert(e)} and \texttt{delete-min} operations) using a data structure called the binary heap and achieved the following guarantees:

- \(O(n)\) space consumption
- \(O(\log n)\) insertion time
- \(O(\log n)\) delete-min time

In this tutorial, we will try to enhance our understanding of the binary heap through some examples and exercises.
Example on Insertion

Assume that we want to insert 12 into the following binary heap. First, add 12 as a leaf, making sure that we still have a complete binary tree.
Then we fix the violations caused by this newly added element.

No more violations, insertion complete. An insertion can be processed in $O(\log n)$ time.
Example on Delete-min

Assume that we want to perform delete-min from this binary heap below:

First, find the rightmost leaf at the bottom level, namely, 37.
Example on Delete-min

Remove this leaf, but place the value 37 in the root.
Example on Delete-min

Then we fix the violations caused by 37.

No more violations, delete-min complete. A delete-min can be processed in $O(\log n)$ time.
Problem

Suppose that we have $k$ sorted arrays (in ascending order) $A_1, A_2, \cdots, A_k$ of integers. Let $n$ be the total number of integers in those arrays. Describe an algorithm to produce an array that sorts all the $n$ integers in ascending order in $O(n \log k)$ time.

Example

Suppose that $k = 3$, and the 3 arrays are as follows:

$A_1$: 2 23 32 35 37 \hspace{1cm} A_2$: 5 10 \hspace{1cm} A_3$: 33 58 82

Then you should produce an array $B$ as below in $O(n \log k)$ time.

$B$: 2 5 10 23 32 33 35 37 58 82
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Solution

Insert the smallest elements of each array into a binary heap $H$. This takes $O(k \log k)$ time. Then, repeat the following until $H$ is empty:

- Perform a `delete-min`. Let $e$ be the element fetched.
- Append $e$ to the output array.
- If $e$ comes from $A_i$ (for some $i$), obtain the next element from $A_i$, and insert it into $H$. If $A_i$ has been exhausted, then do nothing.
Example

Suppose that $k = 3$, and the 3 arrays are as follows:

$A_1$: 2 23 32 35 37  
$A_2$: 5 10  
$A_3$: 33 58 82

First, we insert the smallest elements of each array into a binary heap $H$:

```
   2
  / 
5   33
```

Initially, the output array $B$ is empty.

$B$:          
Example

\[ A_1: 2 \ 23 \ 32 \ 35 \ 37 \quad A_2: 5 \ 10 \quad A_3: 33 \ 58 \ 82 \]

Then, we perform a `delete-min` on \( H \), and fetch \( e = 2 \), then append 2 to the output array \( B \). Since \( e \) comes from \( A_1 \), we obtain the next element 23 from \( A_1 \), and insert it into \( H \).

Output array \( B \): 2
Example

Perform another delete-min on the new $H$, and fetch $e = 5$, then append 5 to the output array $B$. Since $e$ comes from $A_2$, we obtain the next element 10 from $A_2$, and insert it into $H$.

Output array $B$: 2 5
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$A_1: \begin{array}{c} 2 \ 23 \ 32 \ 35 \ 37 \end{array} \quad A_2: \begin{array}{c} 5 \ 10 \end{array} \quad A_3: \begin{array}{c} 33 \ 58 \ 82 \end{array}$

\[
\begin{array}{c}
32 \\
\ \ \ 33 \quad \Rightarrow \text{delete-min} \quad \Rightarrow \ 33 \quad \Rightarrow \text{insert}(35) \quad \Rightarrow \\
\end{array}
\]

Output array $B: \begin{array}{c} 2 \ 5 \ 10 \ 23 \ 32 \end{array}$

\[
\begin{array}{c}
A_1: \begin{array}{c} 2 \ 23 \ 32 \ 35 \ 37 \end{array} \quad A_2: \begin{array}{c} 5 \ 10 \end{array} \quad A_3: \begin{array}{c} 33 \ 58 \ 82 \end{array}
\end{array}
\]

\[
\begin{array}{c}
33 \\
\ \ \ 35 \quad \Rightarrow \text{delete-min} \quad \Rightarrow \ 35 \quad \Rightarrow \text{insert}(58) \quad \Rightarrow \\
\end{array}
\]

Output array $B: \begin{array}{c} 2 \ 5 \ 10 \ 23 \ 32 \ 33 \end{array}$
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Output array $B$: 

$A_1$: [2 23 32 35 37]  
$A_2$: [5 10]  
$A_3$: [33 58 82]  

35  

58 $\Rightarrow$ delete-min $\Rightarrow$ 58 $\Rightarrow$ insert(37) $\Rightarrow$ 37

Output array $B$: 

$A_1$: [2 23 32 35 37]  
$A_2$: [5 10]  
$A_3$: [33 58 82]  

37  

58 $\Rightarrow$ delete-min $\Rightarrow$ 58 $\Rightarrow$ $A_1$ exhausted $\Rightarrow$ do nothing

Output array $B$: 

$A_1$: [2 23 32 35 37]  
$A_2$: [5 10]  
$A_3$: [33 58 82]  

$A_1$: 2 23 32 35 37  $A_2$: 5 10  $A_3$: 33 58 82

58 $\Rightarrow$ delete-min $\Rightarrow$ insert(82) $\Rightarrow$ 82

Output array $B$: 2 5 10 23 32 33 35 37 58 82

$A_1$: 2 23 32 35 37  $A_2$: 5 10  $A_3$: 33 58 82

82 $\Rightarrow$ delete-min $\Rightarrow$ no more insertions $\Rightarrow$ H is empty $\Rightarrow$ stop

Finally, we produce the output array $B$ with all the $n = 10$ elements sorted in ascending order:

Output array $B$: 2 5 10 23 32 33 35 37 58 82
Cost Analysis

- Insert the smallest elements of each array into a binary Heap $H$ takes $O(k \log k)$ time.
- Each delete-min and insertion require $O(\log k)$ time.
  - Since $H$ has at most $k$ elements.
- At most $n$ delete-min and $n$ insertions.
  - Since those arrays contains $n$ elements in total.

Overall, our algorithms takes $O(k \log k) + n \cdot O(\log k) = O(n \log k)$ time.
Problem

Let $S$ be a dynamic set of integers. At the beginning, $S$ is empty. Then, new integers are added to it one by one, but never deleted. Let $k$ be a fixed integer. Describe an algorithm which achieves the following guarantees:

- **Space consumption** $O(k)$
- **Insert($e$)**: Insert a new element $e$ into $S$, which takes at most $O(\log k)$ time.

Example

Suppose that $k = 3$, and the sequence of integers inserted is $83, 21, 66, 5, 24, 76, 92, 33, 43, \ldots$. Your algorithm must be keeping

$\{83, 66, 24\}$ after the insertion of 24, $\{83, 66, 76\}$ after the insertion of 76, and $\{83, 76, 92\}$ after the insertion of 43.
Solution 1

We maintain a binary heap with \( k \) elements, which obviously consumes \( O(k) \) space.

- First, perform \( k \) insertions to build a binary heap \( H \) rooted at \( r \) on the first inserted \( k \) elements of \( S \), and each insertion takes at most \( O(\log k) \) time.

- For a newly inserted integer \( e \), compare it with the root \( r \) of \( H \):
  - If \( e > r \), replace \( r \) with \( e \), and perform root-fix on \( H \).
    - This takes \( O(\log k) \) time.
  - Otherwise, ignore \( e \).

- Then, at any moment, \( H \) contains the \( k \) largest integers of \( S \).
  - Report-top-\( k \) = Report(\( H \)).
Example

Suppose that the sequence of integers inserted is: 83, 21, 66, 5, 24, 76, 92, 33, 43, ..., and \( k = 3 \).

First of all, build a binary heap \( H \) on the first inserted 3 elements \( \{83, 21, 66\} \):

```
  21
 /   \
83   66
```
Example

Suppose that the sequence of integers inserted is: 83, 21, 66, 5, 24, 76, 92, 33, 43, · · · , and \( k = 3 \).

Next, we perform insertions one by one, and see what will happen on our binary heap \( H \):

\[
\begin{align*}
21 & \quad \Rightarrow \text{insert(5)} \Rightarrow \\
83 & \quad 66 \quad \Rightarrow \\
76 & \quad 83 & \quad 66 \quad \Rightarrow \text{insert(24)} \Rightarrow \\
66 & \quad \Rightarrow \text{insert(76)} \Rightarrow \\
83 & \quad 76 & \quad \cdots \Rightarrow \text{insert(43)} \Rightarrow \\
83 & \quad 92
\end{align*}
\]
Solution 2

We maintain an array $A$ with length $2k$, which obviously consumes $O(k)$ space.

- First, append the first inserted $k$ elements of $S$ to $A$.
- Append the $i$-th ($i > k$) inserted integer of $S$ to $A$. Once $A$ is full, do the following:
  - Perform $k$-selection to find the $k$-th largest element of $A$, denoted by $v$.
  - Remove the elements which are smaller than $v$ from $A$.
  - Rearrange $A$ such that $A[1], A[2], \cdots, A[k]$ contains the $k$-largest element respectively.
  - Report-top-$k$. 
Example

Suppose that the sequence of integers inserted is: 83, 21, 66, 5, 24, 76, 92, 33, 43, \ldots, and $k = 3$.

First of all, create an array $A$ with length $2k = 6$.

\[
A: \quad \boxed{\quad \boxed{\quad \boxed{\quad \boxed{\quad \boxed{\quad}}}}}
\]

Then keep insertion one by one.
Example

\[ S = \{83, 21, 66, 5, 24, 76, 92, 33, 43, \cdots \}, \ k = 3. \]

\[
\begin{array}{c}
83 \\
\Rightarrow 83 21 \\
\Rightarrow 83 21 66 \\
\Rightarrow 83 21 66 5 \\
\Rightarrow 83 21 66 5 24 \\
\Rightarrow 83 21 66 5 24 76 \\
\end{array}
\]

A is full now. We perform \textit{k-selection} to find the 3rd-largest integer, which is 66. Then remove the elements which are smaller than 66 from \(A\):

\[
\begin{array}{c}
83 \\
\Rightarrow 83 66 \\
\Rightarrow 83 66 76 \\
\Rightarrow \text{rearrange} \\
\end{array}
\]

So the top-\(k\) (top-3) elements are \(\{83, 66, 76\}\). We can continue insertion like this.
Cost Analysis

- Append the first inserted elements of $S$ to $A$ takes $O(k)$ time.
- Keep insertion, once $A$ is full, we perform $k$-selection to report top-$k$, which takes $O(k)$ time.
- Remove the elements and rearrange $A$ takes $O(k)$ time.

Overall, our algorithm takes $O(k)$ time. Charge these costs to the $k$ insertions indicated below, each insertion bears $O(1)$ time, and each insertion is only charged once.