More on $k$-selection

CSCI2100 Tutorial 5

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Adapted from the slides of the previous offerings of the course
Introduction

Last week, in the lectures, we have learned the $k$-selection problem and solved it in $O(n)$ expected time by making use of randomization. The $k$-selection algorithm discussed in the class is easy to understand and analyze, but less efficient in practice.

In this tutorial, we will introduce a simpler and faster randomized algorithm (but with a more tedious analysis) and discuss another interesting problem related to $k$-selection.
A ”simpler” randomized algorithm

1. Randomly pick an integer $v$ from $S$.

2. Get the rank of $v$, let it be $r$.

3. if $r = k$, return $v$, otherwise:
   3.1 if $r > k$, produce an array $S'$ containing all the integers of $S$ strictly smaller than $v$. Recurse on $S'$ by finding the $k$-th smallest element in $S'$.
   3.2 if $r < k$, produce an array $S'$ containing all the integers of $S$ strictly larger than $v$. Recurse on $S'$ by finding the $(k - r)$-th smallest element in $S'$.
Example

Consider that we want to find the 10-th smallest element from a set $S$ of 12 elements:

\[
\begin{array}{cccccccccccc}
17 & 26 & 38 & 28 & 41 & 72 & 83 & 88 & 5 & 9 & 12 & 35
\end{array}
\]

Suppose that the $\nu$ we randomly choose is 28, whose rank is 6. Since $6 < 10$, we generate an array $S'$ with only the elements larger than 28:

\[
\begin{array}{cccccccc}
38 & 41 & 72 & 83 & 88 & 35
\end{array}
\]

Then we can just recurse by finding the 4-th ($k - r = 10 - 6 = 4$) smallest element in this array $S'$. 

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The above algorithm is procedurally simpler than the one we taught in the class, and is faster in practice too. It, however, is less interesting in two ways:

1. Its analysis is more complicated (in the mundane way).
2. It does not illustrate the ”if-failed-then-repeat” technique.
**Problem:** Let $X[1...n]$ and $Y[1..m]$ be two arrays, both sorted in ascending order. We want to find the $k$-th smallest of the $n + m$ elements where $1 \leq k \leq n + m$. Our algorithm has to end in $O(\log n + \log m)$ time.

**Example:** $X$: $2\ 3\ 6\ 7\ 9\ 12\quad Y$: $1\ 4\ 8\ 10\ 11$

Suppose $k = 5$, then our algorithm should output 6, since the final sorted array is:

```
1\ 2\ 3\ 4\ 6\ 7\ 8\ 9\ 10\ 11\ 12
```
Solution

We solve this problem by recursion.

Base case

The base case happens when either \( n \) or \( m \) is 1. Without loss of generality, assume that \( m = 1 \) (Otherwise, swap the roles of \( X \) and \( Y \)).

- If \( k = n + 1 \), then return \( \max\{X[n], Y[1]\} \).
- Otherwise (i.e., \( k \leq n \)):
  - If \( X[k] < Y[1] \), then return \( X[k] \).
  - Otherwise, return \( \max\{X[k - 1], Y[1]\} \).

Obviously, the base case can be solved in \( O(1) \) time.
Reduce case

Take:

1. The median element $u$ of $X$, namely, $u = X[s]$ where $s = \lfloor n/2 \rfloor$
2. The median element $v$ of $Y$, namely, $v = Y[t]$ where $t = \lfloor m/2 \rfloor$

Without loss of generality, we assume $v \leq u$ (Otherwise, swap the roles of $X$ and $Y$). We distinguish two cases:

- **Case 1:** $s + t \geq k$: None of the elements in $X[s + 1, \ldots n]$ can possibly be the result. We recurse by searching for the $k$-th smallest element of the $s + m$ elements in $X[1\ldots s]$ and $Y[1\ldots m]$.

- **Case 2:** $s + t < k$: None of the elements in $Y[1, \ldots t]$ can possibly be the result. We recurse by searching for the $(k - t)$-th smallest element of the $n + m - t$ elements in $X[1\ldots n]$ and $Y[t + 1\ldots m]$.

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Example

Input $X: \begin{bmatrix} 2 & 8 & 11 & 17 & 20 & 33 & 35 \end{bmatrix}$  
$Y: \begin{bmatrix} 1 & 4 & 7 & 28 & 30 & 43 \end{bmatrix}$  
$k = 5$

Where $n = 7$, $m = 6$, $s = \left\lfloor n/2 \right\rfloor = 3$, $t = \left\lfloor m/2 \right\rfloor = 3$.

We take $u = X[s] = 11$, $v = Y[t] = 7$, and $u > v$.

Since $k = 5$, $s + t = 6 > k$, which follows case 1, then none of the elements in $X[4, \ldots, 7]$ can possibly be the result. We recurse by searching for the 5-th smallest element of the 9 elements in $X[1, \ldots, 3]$ and $Y[1, \ldots, 6]$, i.e.

New Input 1  
$Y: \begin{bmatrix} 2 & 8 & 11 \end{bmatrix}$  
$X: \begin{bmatrix} 1 & 4 & 7 & 28 & 30 & 43 \end{bmatrix}$  
$k = 5$
Example

New Input 1  \( Y: 2 \ 8 \ 11 \)  \( X: 1 \ 4 \ 7 \ 28 \ 30 \ 43 \)  \( k = 5 \)

Where \( n = 6, m = 3, s = \lfloor n/2 \rfloor = 3, t = \lfloor m/2 \rfloor = 1. \)

We take \( u = X[s] = 7, v = Y[t] = 2, \) and \( u > v. \)

Since \( k = 5, s + t = 4 < k, \) which follows case 2, then \( Y[1] \) cannot possibly be the result. We recurse by searching for the \( 5 - 1 = 4 \)-th smallest element of the 8 elements in \( X[1...6] \) and \( Y[2...3], \) i.e.

New Input 2  \( X: 8 \ 11 \)  \( Y: 1 \ 4 \ 7 \ 28 \ 30 \ 43 \)  \( k = 4 \)
Example

New Input 2

\[
X: \begin{bmatrix} 8 & 11 \end{bmatrix} \quad Y: \begin{bmatrix} 1 & 4 & 7 & 28 & 30 & 43 \end{bmatrix} \quad k = 4
\]

Where \( n = 2 \), \( m = 6 \), \( s = \lfloor n/2 \rfloor = 1 \), \( t = \lfloor m/2 \rfloor = 3 \).

We take \( u = X[s] = 8 \), \( v = Y[t] = 7 \), and \( u > v \).

Since \( k = 4 \), \( s + t = 4 = k \), which follows case 1, then \( X[2] \) cannot possibly be the result. We recurse by searching for the 4-th smallest element of the 7 elements in \( X[1] \) and \( Y[1...6] \), i.e.

New Input 3

\[
Y: \begin{bmatrix} 8 \end{bmatrix} \quad X: \begin{bmatrix} 1 & 4 & 7 & 28 & 30 & 43 \end{bmatrix} \quad k = 4
\]

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New Input 3  \[ Y: \begin{array}{c}
8 \\
\end{array} \quad X: \begin{array}{cccccc}
1 & 4 & 7 & 28 & 30 & 43 \\
\end{array} \quad k = 4 \]

This comes to be the base case, since \( k = 4 < n = 6 \), \( X[k] = X[4] = 28 > Y[1] \), we return \( \max\{X[k - 1], Y[1]\} = 8 \).
Cost Analysis

From the above example, we can see that for each recursion, we shrink either $X$ or $Y$ by half. Overall, the above shrinking can happen at most $\log_2 m + \log_2 n$ times before reaching the base case.

It thus follows that the entire algorithm finishes in $O(\log n + \log m)$ time.