Applications of the Binary Search Tree

Shangqi Lu

Department of Computer Science and Engineering
The Chinese University of Hong Kong

Adapted from the slides of the previous offerings of the course
Recall

A binary search tree (BST) on a set $S$ of $n$ integers is a binary tree $T$ satisfying all the following requirements:

- $T$ has $n$ nodes.
- Each node $u$ in $T$ stores a distinct integer in $S$, which is called the key of $u$.
- For every internal $u$, it holds that:
  - The key of $u$ is larger than all the keys in the left subtree of $u$.
  - The key of $u$ is smaller than all the keys in the right subtree of $u$. 
Example

Two possible BSTs on $S = \{3, 11, 12, 15, 18, 29, 40, 41, 47, 68, 71, 92\}$:
Recall

A binary tree $T$ is balanced if the following holds on every internal node $u$ of $T$:

- The height of the left subtree of $u$ differs from that of the right subtree of $u$ by at most 1.
The BST on the left is balanced, while the one on the right is not.
Let $S$ be a set of integers. A predecessor query for a given integer $q$ is to find its predecessor in $S$, which is the largest integer in $S$ that does not exceed $q$. 
Example

Suppose that $S = \{3, 11, 12, 15, 18, 29, 40, 41, 47, 68, 71, 92\}$ and we have a balanced BST $T$ on $S$:

We want to find the predecessor of $q = 42$ in $S$. 
Example

Predecessor query for \( q = 42 \):

- Initialize \( p = -\infty \).
- Initialize \( u \leftarrow \) the root of \( T \).
- Now \( u.key = 40 \) and \( p = -\infty \).
- Since \( u.key < q \), the predecessor of \( q \) must be either \( u \) or some node in the right subtree of \( u \).
- Set \( p = 40 \) and \( u \leftarrow \) the right child of \( u \).
Example

Predecessor query for \( q = 42 \):

- Since \( u \).key > \( q \), the predecessor of \( q \) must be either \( p \) or some node in the left subtree of \( u \).
- Set \( u \leftarrow \) the left child of \( u \).
Example

Predecessor query for $q = 42$:

- Since $u.key < q$, the predecessor of $q$ must be either $u$ or some node in the right subtree of $u$.
- Set $p = 41$ and $u ←$ the right child of $u$. 

![Binary Search Tree Diagram]

CSCI2100, CUHK  Applications of the Binary Search Tree
Example

Predecessor query for $q = 42$:

- Since $u.key > q$, the predecessor of $q$ must be either $p$ or some node in the left subtree of $u$.
- Set $u \leftarrow$ the left child of $u$.
- Since $u$ is nil now, return $p = 41$ as the predecessor of $q$ in $S$. 

![Binary Search Tree Diagram]
**Successor Query**

Let $S$ be a set of integers. A successor query for a given integer $q$ is to find its successor in $S$, which is the smallest integer in $S$ that is no smaller than $q$. 
Successor query for $q = 17$ on $S$:

- Initialize $p = \infty$.
- Initialize $u \leftarrow$ the root of $T$.
- Now $u.key = 40$ and $p = \infty$.
- Since $u.key > q$, the successor of $q$ must be either $u$ or some node in the left subtree of $u$.
- Set $p = 40$ and $u \leftarrow$ the left child of $u$. 
Example

Successor query for $q = 17$ on $S$:

- Since $u.key < q$, the successor of $q$ must be either $p$ or some node in the right subtree of $u$.
- Set $u \leftarrow$ the right child of $u$. 

![Binary Search Tree Diagram](image-url)
Example

Successor query for $q = 17$ on $S$:

- Since $u\text{.key} > q$, the successor of $q$ must be either $u$ or some node in the left subtree of $u$.
- Set $p = 29$ and $u \leftarrow$ the left child of $u$. 
Example

Successor query for $q = 17$ on $S$:

- Since $u.key > q$, the successor of $q$ must be either $u$ or some node in the left subtree of $u$.
- Set $p = 18$ and $u \leftarrow$ the left child of $u$.
- Since $u$ is nil now, return $p = 18$ as the successor of $q$ in $S$. 
In the following, we will discuss how to construct a balanced BST $T$ on a given sorted set $S$ of $n$ integers in $O(n)$ time.
Construction of a Balanced BST

- **Observation 1:** The subtree of any node in a balanced BST is also a balanced BST.

- **Observation 2:** A BST of \( n \) nodes constructed by the following form:

\[
\text{balanced BST of } \left\lfloor \frac{n-1}{2} \right\rfloor \text{ nodes } \quad \text{balanced BST of } \left\lceil \frac{n-1}{2} \right\rceil \text{ nodes}
\]

is a balanced BST (think: why?).
Construction of a Balanced BST

Assume that the $S$ of $n$ integers is stored in an array $A$, the array is sorted.
A balanced BST on $S$ can be constructed as follows:

- **Base Case:**
  - If $n = 0$, return nil.
  - If $n = 1$, create a node $u$ with key $A[1]$ and return the pointer of $u$ as the root of a balanced BST on $A$. 
Construction of a Balanced BST

- **Inductive Case:**
  - Pick the median of $A$ (i.e., $A[\lfloor \frac{n+1}{2} \rfloor]$) and create a node $u$ for it.
  - Recursively construct a balanced BST on the portion of $A$ positioned before the median, and set its root as the left child of $u$.
  - Recursively construct a balanced BST on the portion of $A$ positioned after the median, and set its root as the right child of $u$.
  - Return the pointer of $u$. 

![Diagram of a balanced BST construction](attachment:image.png)
Construction of a Balanced BST

Let $f(n)$ be the maximum running time for constructing a balanced BST from an array of length $n$. Without loss of generality, suppose that $n$ is a power of 2. We have:

\[
  f(1) = O(1) \\
  f(n) = O(1) + 2 \cdot f(n/2)
\]

Solving the recurrence gives $f(n) = O(n)$. 

Example

Let us construct a balanced BST $T$ on a sorted set $S = \{3, 11, 12, 15, 18, 29, 40, 41, 47, 68, 71, 92\}$ by the above algorithm. Suppose that $S$ is stored in an array $A$ of length 12.
Let $S$ be a set of $n$ integers. Given two integers $a$ and $b$ such that $a \leq b$. Find the number of integers in $S$ which are in the range of $[a, b]$.

In the following, we will discuss how to augment a balanced BST on $S$ to achieve:

- $O(n)$ space consumption,
- $O(\log n)$ time for each query.
Range Count Problem

Augment a balanced BST $T$ on $S$ by storing one additional information in each node $u$ that is:

- the number of nodes in the subtree of $u$.

For example,
Range Count Problem

Define a concept first.

- Lowest Common Ancestor: Let $t$ be the root. The lowest common ancestor of nodes $v_1$ and $v_2$ is the **lowest node** that is on both of the paths $P(t, v_1)$ and $P(t, v_2)$.

For example, the lowest common ancestor of node with key 3 and node with key 15 is the node with key 12.

![Binary Search Tree Diagram]

- cnt = 12
- cnt = 5
- cnt = 2
- cnt = 2
- cnt = 2
- cnt = 3
- cnt = 6
- cnt = 1
- cnt = 1
- cnt = 1
- cnt = 1
- cnt = 1
Range Count Problem

For a range $[2, 48]$, let $s$ be the successor of 2, $p$ the predecessor of 48 and $u$ the lowest common ancestor of $s$ and $p$. Initialize a count $c = 1$ (since $u$ is within the range)
Range Count Problem

Traverse the path from $u$’s left child to $s$. For every node $v$ being visited, if $v$.key $\geq$ 2:

- $c += 1$
- $c +=$ the counter of $v$’s right child

$C$ is incremented by $1 + 2$. 
Range Count Problem

Traverse the path from \( u \)'s left child to \( s \). For every node \( v \) being visited, if \( v.key \geq 2 \):

- \( c += 1 \)
- \( c += \) the counter of \( v \)'s right child

\[ C \] is incremented by \( 1 + 1 \).
Range Count Problem

Traverse the path from $u$’s right child to $p$. For every node $v$ being visited, if $v$.key $\leq$ 48:

- $c += 1$
- $c +=$ the counter of $v$’s left child

$C$ is incremented by $1 + 2$. Finally, $c$ becomes 9.
Range Count Problem

We walked through two paths, at most $\log_2 n$ nodes in each path. For each node visited, we perform constant-time operations, which takes $O(1)$.

Time complexity: $O(\log n)$
Range Count Problem

A simpler solution without using a binary search tree

- Use binary search algorithm to find the successor $s$ of $a$
- Use binary search algorithm to find the predecessor $p$ of $b$
- Let $l$ be the index of $s$, then let $u$ be the index $p$
- Return $u - l + 1$

The above algorithm uses two binary search, the time complexity is $O(\log n)$. 
Range Count Problem

Why don’t we just use the simpler solution?
In practice, we may need to update (insert or delete) the elements in S.

Simpler Solution:
- Need to sort S after each update.
- Cost for each update: $O(n \log n)$

Solution with BST:
- Need to insert or delete a node in the BST.
- Cost for each update: $O(\log n)$

That’s why we prefer the BST solution in practice.