More on “Dynamic Array and Amortized Analysis”

CSCI2100 Tutorial 6

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Introduction

In the previous lectures, we have introduced the dynamic array problem and solved it by making use of some clever tricks which allow us to perform $n$ operations in $O(n)$ time, namely, each operation takes $O(1)$ amortized time. We also implemented the data structure stack by exploiting dynamic array.

In this tutorial, we will introduce a new version of dynamic array with smaller space consumption, while each operation still costs $O(1)$ amortized time. We will also try to implement another data structure – the queue – with dynamic array.
Recap: Dynamic Array Problem

Let $S$ be a multi-set of integers that grows with time. At the beginning, $S$ is empty. Over time, the integers of $S$ are added by the following operation:

- **insert($e$):** which adds an integer $e$ into $S$.

At any moment, let $n$ be the number of elements in $S$. We want to store all the elements of $S$ in an array $A$ satisfying:

- **1.** $A$ has length $O(n)$
- **2.** If an integer $x$ was the $i$-th ($i \geq 1$) inserted, then $A[i] = x$ (i.e., $x$ is at the $i$-th position of the array).
Recall that, while performing insertions to a dynamic array $A$, once $A$ is full, we expand $A$ by doubling the current length. We proved that each insertion costs $O(1)$ amortized time and that the space consumption is $O(n)$ at any moment.

In fact, it is not necessary to restrict the expansion to doubling. In the following, we will show a new version of dynamic array which expands the length of $A$ to $1.5n$ once $A$ is full, while each operation still costs $O(1)$ amortized time.
Dynamic Array – A New Version

Perform \texttt{insert}(e) as follows:

- If \( n = 0 \), then set \( n \) to 1. Initialize an array \( A \) with length 2, containing just \( e \) itself.
- Otherwise (i.e., \( n \geq 1 \)), append \( e \) to \( A \), and increase \( n \) by 1. If \( A \) is full, do the following:
  - Initialize an array \( A' \) of length \( \lceil 1.5n \rceil \).
  - Copy all the \( n \) elements of \( A \) over to \( A' \).
  - Destroy \( A \), and replace it with \( A' \).
Example

\[ n = 1 \]

\[ n = 2 \]

\[ n = 3 \]

\[ n = 4 \]

\[ n = 5 \]

\[ n = 6 \]

\[ n = 8 \]
Lemma: When $n \geq 15$, at least $n/4$ elements must have been inserted since the last expansion.

Proof: Let $x$ be the number of elements when the last expansion happened. Hence, $n = \lceil 1.5x \rceil$, meaning that $n - x$ elements have been inserted since the last expansion. It suffices to prove $n - x \geq n/4$ when $n \geq 15$. Towards this purpose, since $n - x \geq 1.5x - x = 0.5x$, it suffices to prove:

$$0.5x \geq \frac{n}{4} = \frac{\lceil 1.5x \rceil}{4}$$

$$\iff 2x \geq \lceil 1.5x \rceil$$

whose correctness can be easily verified for $n \geq 15$. \qed
Cost Analysis

Suppose that the array expansion occurs when \( A \) is full with \( n \) elements, and that expansion takes \( c \cdot n \) time. When \( n \leq 15 \), \( cn = O(1) \). For \( n > 15 \),

- There were \( n/4 \) insertions have taken place since the previous expansion.
- Each of those insertions bears additional \( \frac{cn}{n/4} = 4c = O(1) \) cost.
The Stack-with-Array Problem

Push

We can perform `push(e)` in the same way as an insertion in the dynamic array problem.

Pop

We say that $A$ is sparse if its length is at least 2, and the number of integers therein drops below $4/9$ of its length.

Perform pop as follows:

- Return the last element of $A$, and decrease $n$ by 1. If $A$ is sparse, shrink the array as follows:
  - Initialize an array $A'$ of length $\lceil 1.5n \rceil$.
  - Copy all the elements of $A$ over to $A'$.
  - Destroy $A$, and replace it with $A'$. 
Example

\[ n = 8, \text{ pop} \]

\[ n = 7, \text{ pop} \]

......

\[ n = 5, \text{ pop} \]

......

\[ n = 3, \text{ pop} \]
Cost Analysis

The analysis follows the same ideas explained in the lecture. The crux is to show that, when an “overhaul” (i.e., expansion/shrinking) happens, $\Omega(n)$ operations must have occurred since the last overhaul. As each overhaul takes $O(n)$ time, each of those operations is amortized $O(1)$ time.
The Queue-with-Array Problem

Let $S$ be a multi-set integers that grows with time. At the beginning, $S$ is empty. We must support the following queue operations:

- **En-queue**($e$): Inserts an integer $e$ into $S$.
- **De-queue**: Removes the least recently inserted element from $S$.

At any moment, let $m$ be the number of elements in $S$. We want to store all the elements of $S$ in an array $A$ satisfying:

1. $A$ has length $O(m)$.

We will denote by $n$ the number of operations processed so far.
The Queue-with-Array Problem

We will explain how to maintain a dynamic array that ensures minimum occupancy of 50%. You may apply the techniques explained earlier to increase the minimum occupancy at the tradeoff of higher amortized update cost.
The Queue-with-Array Problem

**En-queue**

Perform \texttt{en-queue(e)} as follows:

- If \( m = 0 \), then set \( m \) to 1. Initialize an array \( A \) with length 2, containing just \( e \) itself.

- Otherwise (i.e., \( m \geq 1 \)), append \( e \) to \( A \), and increase \( m \) by 1. If \( A \) is full, do the following:
  
  - Initialize an array \( A' \) of length \( 2m \).
  - Copy all the \( m \) elements of \( A \) over to \( A' \).
  - Destroy \( A \), and replace it with \( A' \).
The Queue-with-Array Problem

De-queue

Perform de-queue as follows:

- Return the first element of $A$, and decrease $m$ by 1. If $A$ is sparse, shrink the array as follows:
  - Initialize an array $A'$ of length $2m$.
  - Copy all the elements of $A$ over to $A'$.
  - Destroy $A$, and replace it with $A'$.

We say that $A$ is **sparse** if the number of integers therein is equal to $1/4$ of its length.
Next, we use the algorithm to perform 11 en-queues and 9 de-queues on an initially empty queue.

\[
\begin{align*}
  n &= 1, \text{ en-queue} \\
  n &= 2, \text{ en-queue} \\
  n &= 4, \text{ en-queue} \\
  n &= 8, \text{ en-queue} \\
  n &= 11, \text{ en-queue}
\end{align*}
\]
Example

\[ n = 12, \text{ de-queue} \]

[\[
\begin{array}{cccccccccccc}
\text{orange} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} \\
\end{array}
\]

......

\[ n = 17, \text{ de-queue} \]

[\[
\begin{array}{cccccccccccc}
\text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} \\
\end{array}
\]

\[ n = 18, \text{ de-queue} \]

[\[
\begin{array}{cccccccccccc}
\text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} \\
\end{array}
\]

\[ n = 19, \text{ de-queue} \]

[\[
\begin{array}{cccccccccccc}
\text{orange} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} & \text{green} \\
\end{array}
\]

\[ n = 20, \text{ de-queue} \]

[\[
\begin{array}{cccccccccccc}
\text{green} & \text{green} & \text{green} \\
\end{array}
\]
The cost of expansion is at most $c_1 \cdot s$ for some constant $c_1$. By charging the cost over the $s/2$ en-queue operations as indicated above, each operation bears at most $2c_1$ cost.

The cost of shrinking is at most $c_2 \cdot s$ for some constant $c_2$. By charging the cost over the $s/4$ de-queue operations as indicated above, each operation bears at most $4c_2$ cost.

Hence, performing any sequence of operations using $O(n)$ time in total, and each operation (either an en-queue or a de-queue) is guaranteed to cost $O(1)$ amortized time.