Exercises on “the Growth of Functions”

CSCI2100 Tutorial 2

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Introduction

Last week, we have learned two different ways to decide whether one function $f(n)$ grows faster than another $g(n)$:

- The first one achieves the purpose by finding appropriate “constants $c_1, c_2$”.
- The second is by inspecting the ratio $\frac{f(n)}{g(n)}$ as $n \to \infty$.

In this tutorial, we will apply both methods through some exercises.
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).
Exercise 1

Let $f(n) = 10n + 5$ and $g(n) = n^2$. Prove $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

Direction 1: Constant Finding

$f(n) = O(g(n))$, if there exist two positive constants $c_1$ and $c_2$ such that $f(n) \leq c_1 \cdot g(n)$ holds for all $n \geq c_2$. 
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 1: Constant Finding

Proof of \( f(n) = O(g(n)) \)

Our mission is to find \( c_1, c_2 \) to make \( f(n) \leq c_1 \cdot g(n) \) hold for all \( n \geq c_2 \). Remember: we do not need to find the smallest \( c_1, c_2 \); instead, it suffices to obtain any \( c_1, c_2 \) that can do the job. Indeed, we will often go for some “easy” selections that can simplify derivation.
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 1: Constant Finding

Proof of \( f(n) = O(g(n)) \)

\[
10n + 5 \leq c_1 \cdot n^2 \\
\iff 5(2n + 1) \leq c_1 \cdot n^2 \quad \text{(let } c_1 = 5) \\
\iff 2n + 1 \leq n^2 \\
\iff 2 \leq (n - 1)^2 \\
\iff 3 \leq n
\]

Hence, it suffices to set \( c_2 = 3 \).
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 1: Constant Finding

Proof of \( g(n) \neq O(f(n)) \)

Let us prove this by contradiction. Suppose, on the contrary, that \( g(n) = O(f(n)) \). This means the existence of constants \( c_1, c_2 \) such that, we have for all \( n \geq c_2 \)

\[
\begin{align*}
n^2 &\leq c_1 \cdot (10n + 5) \\
\Rightarrow \quad n^2 &\leq c_1 \cdot 20n \\
\iff \quad n &\leq 20c_1
\end{align*}
\]

which cannot always hold for all \( n \geq c_2 \). This completes the proof.
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 2: Inspecting \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \)

Proof of \( f(n) = O(g(n)) \)

\[
\lim_{n \to \infty} \frac{10n + 5}{n^2} = \lim_{n \to \infty} \frac{10 + 5/n}{n} = 0.
\]

Hence, \( f(n) = O(g(n)) \).
Exercise 1

Let \( f(n) = 10n + 5 \) and \( g(n) = n^2 \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).

Direction 2: Inspecting \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \)

Proof of \( g(n) \neq O(f(n)) \)

\[
\lim_{n \to \infty} \frac{n^2}{10n + 5} = \infty.
\]

Hence, \( g(n) \neq O(f(n)) \).
Exercise 2

Let \( f(n) = 5 \log_2 n \) and \( g(n) = \sqrt{n} \). Prove \( f(n) = O(g(n)) \) and \( g(n) \neq O(f(n)) \).
Direction 1: Constant Finding

Proof of $f(n) = O(g(n))$

Setting $c_1 = 5$, we want:

$$5 \log_2 n \leq 5 \cdot \sqrt{n}$$

$$\Leftrightarrow \log_2 n \leq \sqrt{n}$$

Hence, it suffices to set $c_2 = 64$. 
Direction 1: Constant Finding

Proof of $g(n) \neq O(f(n))$

We prove this by contradiction. Suppose that $g(n) = O(f(n))$. It implies that there exist constants $c_1, c_2$ such that for all $n \geq c_2$, we have

$$\sqrt{n} \leq c_1 \cdot 5 \cdot \log_2 n$$

$$\iff \frac{\sqrt{n}}{\log_2 n} \leq 5c_1$$

which cannot always hold for all $n \geq c_2$. This completes the proof.
Direction 2: Inspecting \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \)

Proof of \( f(n) = O(g(n)) \)

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{5 \log_2 n}{\sqrt{n}} = 0.
\]
Thus, we have \( f(n) = O(g(n)) \).

Proof of \( g(n) \neq O(f(n)) \).

\[
\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{\sqrt{n}}{5 \log_2 n} = \infty.
\]
Hence, \( g(n) \neq O(f(n)) \).
Exercise 3

Given that $10n + 5 = O(n^2)$ and $5 \log_2 n = O(\sqrt{n})$, prove $10n + 5 + 5 \log_2 n = O(n^2 + \sqrt{n})$. 
**Direction 1: Constant Finding**

Since $10n + 5 = O(n^2)$ implies the existence of constants $c_1$ and $c_2$ such that $10n + 5 \leq c_1 \cdot n^2$ holds for all $n \geq c_2$.

Similarly, $5 \log_2 n = O(\sqrt{n})$ means there exist two constants $c_1'$ and $c_2'$ which make $5 \log_2 n \leq c_1' \cdot \sqrt{n}$ hold for all $n \geq c_2'$.

Thus:

$$10n + 5 + 5 \log_2 n \leq c_1 n^2 + c_1' \sqrt{n} \leq \max\{c_1, c_1'\} \cdot (n^2 + \sqrt{n})$$

holds for all $n \geq \max\{c_2, c_2'\}$.

Therefore, $10n + 5 + 5 \log_2 n = O(n^2 + \sqrt{n})$. 
Direction 2: Inspecting $\lim_{n \to \infty} \frac{f(n)}{g(n)}$

Since $10n + 5 = O(n^2)$, we have $\lim_{n \to \infty} \frac{10n + 5}{n^2} = c$, where $c$ is some constant.

Similarly, $5 \log_2 n = O(\sqrt{n})$ indicates that $\lim_{n \to \infty} \frac{5 \log_2 n}{\sqrt{n}} = c'$, where $c'$ is some constant.

Both of the above imply that:

$$\lim_{n \to \infty} \frac{10n + 5 + 5 \log_2 n}{n^2 + \sqrt{n}} = \lim_{n \to \infty} \frac{10n + 5}{n^2 + \sqrt{n}} + \lim_{n \to \infty} \frac{5 \log_2 n}{n^2 + \sqrt{n}}$$
$$\leq \lim_{n \to \infty} \frac{10n + 5}{n^2} + \lim_{n \to \infty} \frac{5 \log_2 n}{\sqrt{n}}$$
$$= c + c'.$$

Therefore, $10n + 5 + 5 \log_2 n = O(n^2 + \sqrt{n})$. 
Exercise 4

Consider functions of $n$: $f_1(n)$, $f_2(n)$, $g_1(n)$ and $g_2(n)$ such that:

$$f_1(n) = O(g_1(n)) \text{ and } f_2(n) = O(g_2(n))$$

Prove $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$. 
Direction 1: Constant Finding

Since \( f_1(n) = O(g_1(n)) \), there exist constants \( c_1 \) and \( c_2 \) such that \( f_1(n) \leq c_1 \cdot g_1(n) \) holds for all \( n \geq c_2 \).

Similarly, \( f_2(n) = O(g_2(n)) \) implies the existence of constants \( c'_1 \) and \( c'_2 \) such that \( f_2(n) \leq c'_1 \cdot g_2(n) \) holds for all \( n \geq c'_2 \).

Thus:

\[
f_1(n) + f_2(n) \leq c_1 \cdot g_1(n) + c'_1 \cdot g_2(n) \leq \max\{c_1, c'_1\} \cdot (g_1(n) + g_2(n))
\]

for all \( n \geq \max\{c_2, c'_2\} \).

Therefore, \( f_1(n) + f_2(n) = O(g_1(n) + g_2(n)) \).
Direction 2: Inspecting $\lim_{n \to \infty} \frac{f(n)}{g(n)}$

Since $f_1(n) = O(g_1(n))$, we have $\lim_{n \to \infty} \frac{f_1(n)}{g_1(n)} = c$ for some constant $c$.

Similarly, $f_2(n) = O(g_2(n))$ indicates $\lim_{n \to \infty} \frac{f_2(n)}{g_2(n)} = c'$ for some constant $c'$.

This leads to:

$$\lim_{n \to \infty} \frac{f_1(n) + f_2(n)}{g_1(n) + g_2(n)} = \lim_{n \to \infty} \frac{f_1(n)}{g_1(n) + g_2(n)} + \lim_{n \to \infty} \frac{f_2(n)}{g_1(n) + g_2(n)}$$

$$\leq \lim_{n \to \infty} \frac{f_1(n)}{g_1(n)} + \lim_{n \to \infty} \frac{f_2(n)}{g_2(n)}$$

$$\leq c + c'.$$

Therefore, $f_1(n) + f_2(n) = O(g_1(n) + g_2(n))$. 
Exercise 5

Given that $100n + \sqrt{n} + \log_2 n = \Theta(n)$, prove

$$\sqrt{100n + \sqrt{n} + \log_2 n} = \Theta(\sqrt{n}).$$
Exercise 5

Given that $100n + \sqrt{n} + \log_2 n = \Theta(n)$, prove $\sqrt{100n + \sqrt{n} + \log_2 n} = \Theta(\sqrt{n})$.

Big-Θ

Let $f(n)$ and $g(n)$ be two functions of $n$.

- If $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then we define $f(n) = \Theta(g(n))$.
- There exist positive constants $c_1$, $c_2$ and $c_3$, such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$, for all $n \geq c_3$. 
Direction 1: Constant Finding

Since $100n + \sqrt{n} + \log_2 n = \Theta(n)$, there exist positive constants $c_1$, $c_2$ and $c_3$ such that $c_1 n \leq 100n + \sqrt{n} + \log_2 n \leq c_2 n$ holds for all $n \geq c_3$.

Thus, for all $n \geq c_3$,

$$\sqrt{c_1 n} \leq \sqrt{100n + \sqrt{n} + \log_2 n} \leq \sqrt{c_2 n} \Rightarrow \sqrt{c_1} \sqrt{n} \leq \sqrt{100n + \sqrt{n} + \log_2 n} \leq \sqrt{c_2} \sqrt{n} \Rightarrow c'_1 \sqrt{n} \leq \sqrt{100n + \sqrt{n} + \log_2 n} \leq c'_2 \sqrt{n}$$

Where $c'_1 = \sqrt{c_1}$, $c'_2 = \sqrt{c_2}$.

Therefore, $\sqrt{100n + \sqrt{n} + \log_2 n} = \Theta(\sqrt{n})$. 

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Direction 2: Inspecting \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \)

Since \( 100n + \sqrt{n} + \log_2 n = \Theta(n) \), we have \( \lim_{n \to \infty} \frac{100n + \sqrt{n} + \log_2 n}{n} = c_1 \) for some constant \( c_1 \), and \( \lim_{n \to \infty} \frac{n}{100n + \sqrt{n} + \log_2 n} = c_2 \) for some constant \( c_2 \).

So, on one hand:

\[
\lim_{n \to \infty} \sqrt{100n + \sqrt{n} + \log_2 n} = \lim_{n \to \infty} \sqrt{\frac{100n + \sqrt{n} + \log_2 n}{n}} = \sqrt{\lim_{n \to \infty} \frac{100n + \sqrt{n} + \log_2 n}{n}} = \sqrt{c_1}
\]

Therefore, \( \sqrt{100n + \sqrt{n} + \log_2 n} = O(\sqrt{n}) \).
Direction 2: Inspecting $\lim_{n \to \infty} \frac{f(n)}{g(n)}$

Since $100n + \sqrt{n} + \log_2 n = \Theta(n)$, we have $\lim_{n \to \infty} \frac{100n + \sqrt{n} + \log_2 n}{n} = c_1$ for some constant $c_1$, and $\lim_{n \to \infty} \frac{n}{100n + \sqrt{n} + \log_2 n} = c_2$ for some constant $c_2$.

On the other hand:

$$
\lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{100n + \sqrt{n} + \log_2 n}} = \lim_{n \to \infty} \frac{n}{\sqrt{100n + \sqrt{n} + \log_2 n}} = \sqrt{\lim_{n \to \infty} \frac{n}{100n + \sqrt{n} + \log_2 n}} = \sqrt{c_2}
$$

Therefore, $\sqrt{n} = O(\sqrt{100n + \sqrt{n} + \log_2 n})$. 

Direction 2: Inspecting \( \lim_{n \to \infty} \frac{f(n)}{g(n)} \)

Since \( 100n + \sqrt{n} + \log_2 n = \Theta(n) \), we have \( \lim_{n \to \infty} \frac{100n + \sqrt{n} + \log_2 n}{n} = c_1 \) for some constant \( c_1 \), and \( \lim_{n \to \infty} \frac{n}{100n + \sqrt{n} + \log_2 n} = c_2 \) for some constant \( c_2 \).

Finally, we have:

\[
\sqrt{100n + \sqrt{n} + \log_2 n} = O(\sqrt{n})
\]

\[
\sqrt{n} = O(\sqrt{100n + \sqrt{n} + \log_2 n})
\]

Therefore, \( \sqrt{100n + \sqrt{n} + \log_2 n} = \Theta(\sqrt{n}) \).
This tutorial gives us some exercises about how to prove the "Big-O" relationship between two functions by using two different methods.

Following the definitions strictly will always lead us to the right proof.