Side Talk: More on Big-O

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In the class, we have learned that, intuitively, \( f(n) = O(g(n)) \) means “function \( f(n) \) grows asymptotically no faster than function \( g(n) \)”. In the next few slides, we will reinforce this understanding from a graphical point of view.
Quadratic vs. Linear

\[ f(n) = n^2 \] and \[ g(n) = 100n. \]

The left figure indicates that \( f(n) \neq O(g(n)) \), while the right one indicates \( g(n) = O(f(n)) \). Think: Why?
Exponential vs. Quadratic

\[ f(n) = 1.1^n \] and \[ g(n) = n^2. \]

The left figure indicates that \( f(n) \neq O(g(n)) \), while the right one indicates \( g(n) = O(f(n)) \).
Polynomial vs. Poly-Logarithmic

\[ f(n) = n^{1.1} \text{ and } g(n) = (\log_2 n)^9. \]

The left figure indicates that \( f(n) \neq O(g(n)) \), while the right one indicates \( g(n) = O(f(n)) \).
An Example of $\Theta$

$f(n) = 10n^2$ and $g(n) = n^2 - \sqrt{n} + (\log_2 n)^3$.

The left figure indicates that $f(n) = O(g(n))$, while the right one indicates $g(n) = O(f(n))$. In other words, $f(n) = \Theta(g(n))$. 
Our final words concern the definition of big-O. Recall that our “official”
definition of $f(n) = O(g(n))$ is:

There is a constant $c_1 > 0$ such that $f(n) \leq c_1 \cdot g(n)$ holds for all
$n$ at least a constant $c_2$.

In the lecture, we also mentioned that $f(n) = O(g(n))$ when
$\lim_{n \to \infty} \frac{f(n)}{g(n)}$ is at most some constant $c$. This provides an alternative
approach to prove the big-O.

However, it must be emphasized that $\lim_{n \to \infty} \frac{f(n)}{g(n)} \leq c$ is only a
necessary condition of big-O, but not a sufficient condition. Why?
Because it is possible that $f(n) = O(g(n))$, and yet, $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ does
not exist! We will see an example in the next slide.
Consider $f(n) = 2^n$. Define $g(n)$ as:

- $g(n) = 2^n$ if $n$ is even;
- $g(n) = 2^{n-1}$ otherwise.

Since $g(n) \leq 2f(n)$ holds for all $n \geq 1$, it holds that $g(n) = O(f(n))$.

However, $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ does not exist, because it keeps jumping between 1 and 2 as $n$ increases!