CSCI2100: Regular Exercise Set 13

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Problem 1 (Correctness of Dijkstra) Prove that Dijkstra’s algorithm correctly computes all the shortest paths from the source vertex.

Solution. Let $s$ be the source vertex. Recall that the algorithm works by repetitively removing the vertex $u$ from $S$ that has the smallest $dist(u)$. We will prove that, when $u$ is removed, $dist(u)$ equals precisely the shortest path distance—denoted as $spdist(u)$—from $s$ to $u$.

We will prove the claim by induction on the sequence of vertices removed. This is obviously true for the first vertex removed, which is $s$ itself with $dist(s) = 0$.

Now consider that we are removing vertex $u$ from $S$, and the claim is true with respect to all the vertices already removed. Consider any shortest path $\pi$ from $s$ to $u$. Let $v$ be the predecessor of $u$ on this path. We will prove that $v$ has already been removed. This will complete the proof because when $v$ is removed, we have:

- $spdist(v) = dist(v)$
- Relaxing the edge $(v, u)$ makes $dist(u) = dist(v) + w(u, v) = spdist(v)$.

We will prove that all the vertices on $\pi$ have been removed (and hence, $v$ as well) at the moment when $u$ is removed. Suppose that this is not true. Let $v'$ be the first vertex (in the direction from $s$ to $u$) on $\pi$ that still remains in $S$. Let $p$ be the predecessor of $v'$ on $\pi$. By the inductive assumption, we know that $dist(p) = spdist(p)$ when $p$ was removed. Hence, after relaxing the edge $(p, v')$, we had $dist(v') = dist(p) + w(p, v') = spdist(v') < dist(u)$. This means that $v'$ should be the next vertex to remove, contradicting that the algorithm has chosen $u$.

Problem 2. Let $S$ be a set of integer pairs of the form $(id, v)$. We will refer to the first field as the id of the pair, and the second as the key of the pair. Design a data structure that supports the following operations:

- Insert: add a new pair $(id, v)$ to $S$ (you can assume that $S$ does not already have a pair with the same id).
- Delete: given an integer $t$, delete the pair $(id, v)$ from $S$ where $t = id$, if such a pair exists.
- DeleteMin: remove from $S$ the pair with the smallest key, and return it.

Your structure must consume $O(n)$ space, and support all operations in $O(\log n)$ time where $n = |S|$.

Solution. Maintain $S$ in two binary search trees $T_1$ and $T_2$, where the pairs are indexed on ids in $T_1$, and on keys in $T_2$. We support the three operations as follows:

- Insert: simply insert the new pair $(id, v)$ into both $T_1$ and $T_2$.
- Delete: first find the pair with id $t$ in $T_1$, from which we know the key $v$ of the pair. Now, delete the pair $(t, v)$ from both $T_1$ and $T_2$.
- DeleteMin: find the pair with the smallest key $v$ from $T_2$ (which can be found by continuously descending into left child nodes). Now we have its id $t$ as well. Remove $(t, v)$ from $T_1$ and $T_2$. 

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Problem 3. Describe how to implement the Dijkstra’s algorithm on a graph \( G = (V,E) \) in \( O((|V| + |E|) \cdot \log |V|) \) time.

Solution. Recall that the algorithm maintains (i) a set \( S \) of vertices at all times, and (ii) an integer value \( \text{dist}(v) \) for each vertex \( v \in S \). Define \( P \) to be the set of \( (v, \text{dist}(v)) \) pairs (one for each \( v \in S \)). We need the following operations on \( P \):

- Insert: add a pair \( (v, \text{dist}(v)) \) to \( P \).
- DecreaseKey: given a vertex \( v \in S \) and an integer \( x < \text{dist}(v) \), update the pair \( (v, \text{dist}(v)) \) to \( (v, x) \) (and thereby, setting \( \text{dist}(v) = x \) in \( P \)).
- DeleteMin: Remove from \( P \) the pair \( (v, \text{dist}(v)) \) with the smallest \( \text{dist}(v) \).

We can store \( P \) in a data structure of Problem 2 which supports all operations in \( O(\log |V|) \) time (note: DecreaseKey can be implemented as a Delete followed by an Insert).

In addition to the above structure, we store all the \( \text{dist}(v) \) values in an array \( A \) of length \( |V| \), so that using the id of a vertex \( v \), we can find its \( \text{dist}(v) \) in constant time.

Now we can implement the algorithm as follows. Initially, insert only \( (s,0) \) into \( P \), where \( s \) is the source vertex. Also, in \( A \), set all the values to \( \infty \), except the cell of \( s \) which equals 0.

Then, we repeat the following until \( P \) is empty:

- Perform a DeleteMin to obtain a pair \( (v, \text{dist}(v)) \).
- For every edge \( (v, u) \), compare \( \text{dist}(u) \) to \( \text{dist}(v) + w(u,v) \). If the latter is smaller, perform a DecreaseKey on vertex \( u \) to set \( \text{dist}(u) = \text{dist}(v) + w(u,v) \), and update the cell of \( u \) in \( A \) with this value as well.

Problem 4. Prove: in a weighted undirected graph \( G = (V,E) \) where all the edges have distinct weights, the minimum spanning tree (MST) is unique.

Solution. We will prove that the tree \( T \) returned by the Prim’s algorithm is the only MST. Set \( n = |V| \). Let \( e_1, e_2, ..., e_{n-1} \) be the sequence of edges that the algorithm adds to \( T \). Suppose, on the contrary, that there is another MST \( T' \). Let \( k \) be the smallest \( i \) such that \( e_i \) is not in \( T' \).

- Case 1: \( k = 1 \). This means that \( e_1 \), which is the edge with the smallest weight, is not in \( T' \). Add \( e_1 \) to \( T' \) to create a cycle, and remove from the cycle the edge with the largest weight. This create another spanning tree whose cost is strictly smaller than \( T' \) (remember: all the edges are distinct), contradicting the fact that \( T' \) is an MST.

- Case 2: \( k > 1 \). Recall that edges \( e_1, e_2, ..., e_{k-1} \) form a tree. Let \( S \) be the set of vertices in this tree. Add \( e_k = \{u,v\} \) into \( T' \) to create a cycle. Suppose \( u \in S \); it follows that \( v \notin S \). Let us walk on the cycle from \( v \), by going into \( S \), traveling within \( S \), and stopping as soon as we exist \( S \). Let \( \{u',v'\} \) be the last edge crossed (namely, one of \( u',v' \) is in \( S \), while the other one is not). By the way Prim’s algorithm runs and the fact that all edges have distinct weights, we know that \( \{u,v\} \) has a smaller weight than \( \{u', v'\} \). Thus, removing \( \{u', v'\} \) from \( T' \) gives spanning tree with strictly smaller cost, which creates a contradiction.

Problem 5. Describe how to implement the Prim’s algorithm on a graph \( G = (V,E) \) in \( O((|V| + |E|) \cdot \log |V|) \) time.
Solution. Remember that the algorithm incrementally grows a tree $T$ which at the end becomes the final minimum spanning tree. Let $S$ be the set of vertices that are currently in $T$. At all times, the algorithm maintains, for every vertex $v \in V \setminus S$, its lightest extension edge $\text{best-ext}(v)$, and the weight of this edge.

To implement this, we maintain a set $P$ of triples, one for every vertex $u \in V \setminus S$. Specifically, the triple of $u$ has the form $(u, v, t)$, indicating that $\text{best-ext}(u)$ is the edge $\{u, v\}$ (i.e., $v \in S$), whose weight is $t$. We need the following operations on $P$:

- Insert: add a triple $(u, v, t)$ to $P$.
- DecreaseKey: given a vertex $v' \in S$ and an extension edge $\{u, v'\}$ (i.e., $u \notin S$), this operation does the following. First, fetch the triple $(u, v, t)$. Then, compare $t$ to the weight $t'$ of $\{u, v'\}$. If $t' < t$, update the triple $(u, v, t)$ to $(u, v', t')$; otherwise, do nothing.
- DeleteMin: Remove from $P$ the triple $(u, v, t)$ with the smallest $t$.

We can store $P$ in a data structure of Problem 2 which supports all operations in $O(\log |V|)$ time (note: DecreaseKey can be implemented as a Delete followed by an Insert). Besides the above structure, we also store an array $A$ of length $|V|$ to so that we can query in constant time, for any vertex $v \in V$, whether $v$ is in $S$ currently.

Now we can implement the algorithm as follows. Let $\{v_1, v_2\}$ be an edge with the smallest weight in $G$. The set $S$ contains only $v_1$ and $v_2$ at this point. For every vertex $u \in V \setminus S$ where $S = \{v_1, v_2\}$, we check whether $u$ has extension edges to $v_1$ and $v_2$. If neither edge exists, insert triple $(u, \text{nil}, \infty)$ to $P$. Otherwise, suppose without loss of generality that $\{u, v_1\}$ is the lighter extension edge of $u$ with weight $t$; insert a triple $(u, v_1, t)$ into $P$.

Repeat the following until $P$ is empty:

- Perform a DeleteMin to obtain a triple $(u, v, t)$.
- Recall that $u$ should be added to $S$, which may need to change the extension edges of some other vertices. To implement this, for every edge $(u, u')$ of $u$ where $u' \notin S$, perform DecreaseKey with $u'$ and $\{u, u'\}$.