ENGG1410 Midterm (105 minutes)

No calculators allowed. Double-sided HANDWRITTEN A4 sized cheat-sheet allowed.
Cheating will be dealt with severely.

Name: Student ID:

1. Let \( f(x, y) = x^2 + y^2 - 2x + 4y + 10 \).

   (a) (4 points) Find the point where \( \nabla f \), the gradient of \( f \), equals 0, the zero vector. Sketch the vector field \( \nabla f \) around this point. (Draw at least 8 arrows.)

   (b) (3 points) For any positive \( c \), find the curve \( C \) such that the length of \( \nabla f \) equals \( c \).

   (c) (3 points) For any point \((x_0, y_0)\), find a vector \( v \) such that the directional derivative of \( f \) at \((x_0, y_0)\) in the direction of \( v \) equals zero.

2. Given a vector field:
   \[ \mathbf{v}(x, y, z) = \begin{bmatrix} 2xy, & x^2 + 2yz, & x + y^2 \end{bmatrix} \]

   (a) (5 points) Let \( g(x, y, z) = x \). Show that \( \text{div}(g \mathbf{v}) = g \text{div}(\mathbf{v}) + \mathbf{v} \cdot \nabla g \).

   (b) (5 points) Prove that there does NOT exist a function \( f(x, y, z) \) such that \( \nabla f(x, y, z) = \mathbf{v}(x, y, z) \).

3. Consider the curve \( C: y^2 = x^3 \), where \( 0 \leq x \leq 4 \) and \( y \leq 0 \).

   (a) (3 points) Find a parametric representation for the curve \( C \).

   (b) (3 points) Find its tangent line at the point \( P : (1, -1) \).

   (c) (4 points) Calculate the length of the curve \( C \).

4. (a) (6 points) Let \( \mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \), \( \mathbf{b} = 2\mathbf{i} + 2\mathbf{j} \), and \( \theta \) be the angle between these two vectors. Calculate \( \mathbf{a} \cdot \mathbf{b}, \mathbf{a} \times \mathbf{b}, \cos \theta \) and \( \sin \theta \).

   (b) (4 points) Find the distance from the point \( q = (4, 3, 2) \) to the line \( l \) which goes through the point \( p = (1, 4, 1) \) and is parallel to the vector \( \mathbf{b} \).

5. Let \( C \) be the triangle constructed from the three points \((0, 0), (a, a), (-a, a)\) with some \( a > 0 \), and traversed counterclockwise. Denote \( \mathbf{F} = [y - \frac{x}{3}y^3 + 6x^2y, 0] \).

   (a) (5 points) Represent \( \oint_C \mathbf{F}(r) \cdot dr \) as a double integral.

   (b) (5 points) For what value of \( a > 0 \) do we have \( \oint_C \mathbf{F}(r) \cdot dr = -1 \)?

6. (a) (5 points) Let \( C \) be the curve \([2t\cos(t)^5, 2t^2\sin(t)^7, 3t] \) with \( t \) from 0 to 2\( \pi \). Calculate \( \int_C (y \, dx + x \, dy + dz) \).

   (b) (5 points) Consider the following two surfaces:
   \[
   x^2 + y^2 - z = 1 \\
   x = 2y.
   \]
   Both points \( p = (0, 0, -1) \) and \( q = (2, 1, 4) \) are on the above surfaces. Let \( C \) be the curve that goes from \( p \) to \( q \) along the intersection of the above surfaces. Calculate \( \int_C (y \, dx - x \, dy + dz) \).
Solutions

1. (a) By direct computation, \( \nabla f = (2x - 2)i + (2y + 4)j \). Therefore \( \nabla f = 0 \) at the point \((1, -2)\).
   The vector field of the gradient function points outward from this point, in a radially symmetric manner.

(b) Since the length of \( \nabla f \) equals \( \sqrt{4(x - 1)^2 + 4(y + 2)^2} \), hence the curve is \( 4(x - 1)^2 + 4(y + 2)^2 = c^2 \).

(c) One can directly observe that the vector \(- (2y_0 + 4)i + (2x_0 - 2)j\) is perpendicular to the vector \((2x_0 - 2)i + (2y_0 + 4)j\). (Alternatively, one can assume the vector as the form \([a, b]\), and then solve the equation \([a, b] \cdot [2x - 2, 2y + 4] = 0\), and choose \(a\) as an arbitrary non-zero value.)

2. (a)

\[
\begin{align*}
LHS &= \text{div}(gv) \\
&= \frac{\partial xv_1}{\partial x} + \frac{\partial xv_2}{\partial y} + \frac{\partial xv_3}{\partial z} \\
&= 4xy + 2xz + 0 \\
&= x(4y + 2z) \\
RHS &= g \text{div}(v) + v \cdot \nabla g \\
&= x \left( \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) + v \cdot [1, 0, 0] \\
&= x(2y + 2z + 0) + 2xy \\
&= x(4y + 2z)
\end{align*}
\]

(b) Consider
\[
\text{curl}(v) = \begin{vmatrix}
i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2xy & x^2 + 2yz & x + y^2
\end{vmatrix}
\]
\[
= \left[ \frac{\partial(x + y^2)}{\partial y} - \frac{\partial(x^2 + 2y - z)}{\partial z} \right] i - \left[ \frac{\partial(x + 2y)}{\partial x} - \frac{\partial(2xy)}{\partial z} \right] j \\
+ \left[ \frac{\partial(x^2 + 2yz)}{\partial x} - \frac{\partial(2xy)}{\partial y} \right] k
\]
\[
= (2y - 2y)i - (1 - 0)j + (2x - 2x)k \\
= -j
\]

Thus, the desired function does not exist for \(v\).
3. (a) Let \( x = t \). Then, \( y = -t^{3/2} \), and a parametric representation of the curve is given by \( r(t) = [t, -t^{3/2}] \), with \( 0 \leq t \leq 4 \). An alternative solution is to let \( x = t^2 \). Then, \( y = -t^3 \), and the parametric representation is given by \( r(t) = [t^2, -t^3] \), with \( 0 \leq t \leq 2 \).

(b) Note that the point \( P \) corresponds to the position vector \( r(1) = [1, -1] \). Since \( r'(t) = [1, -3\sqrt{t}/2] \), we have \( r'(1) = [1, -3/2] \). Therefore, the tangent line at point \( P \) can be expressed as

\[
q(w) = r(1) + wr'(1) = [1 + w, -1 - 3w/2], \quad -\infty < w < +\infty.
\]

If the curve \( C \) is expressed in the alternative way, then the point \( P \) corresponds to the position vector \( r(1) = [1, -1] \). Since \( r'(t) = [2t, -3t^2] \), we have \( r'(1) = [2, -3] \). Therefore, the tangent line at point \( P \) can be expressed as

\[
q(w) = r(1) + wr'(1) = [1 + 2w, -1 - 3w], \quad -\infty < w < +\infty.
\]

(c) According to the formula for curve length, we have

\[
s = \int_0^4 \sqrt{r'(t) \cdot r'(t)} dt = \int_0^4 \sqrt{1 + \frac{9}{4}t} dt
\]

\[
= \frac{4}{9} \int_0^9 \sqrt{u + 1} du = \frac{8}{27} (u + 1)^{9/2} \bigg|_{u=0}^{u=9} = \frac{8}{27} (10\sqrt{10} - 1).
\]

The alternative representation will give the same result after change of variables.

4. (a) \( \mathbf{a} \cdot \mathbf{b} = 3 \times 2 - 1 \times 2 + 1 \times 0 = 4. \)

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  3 & -1 & 1 \\
  2 & 2 & 0
\end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}.
\]

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| \, ||\mathbf{b}||} = \frac{4}{\sqrt{11} \times 8} = \frac{2}{\sqrt{22}}.
\]

\[
\sin \theta = \frac{||\mathbf{a} \times \mathbf{b}||}{||\mathbf{a}|| \, ||\mathbf{b}||} = \frac{\sqrt{72}}{\sqrt{11} \times 8} = \frac{3}{\sqrt{11}}.
\]

(b) \( \mathbf{pq} = [3, -1, 1] = \mathbf{a} \), the distance from the point \( q \) to the line \( l \) is \( ||\mathbf{pq}|| \sin \theta = 3. \)
5. (a) Note that $\frac{\partial F_1}{\partial y} = 1 - 2y^2 + 6x^2$, $\frac{\partial F_2}{\partial x} = 0$. By Green’s Theorem, we have

$$\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \iint_D (-1 + 2y^2 - 6x^2)dxdy,$$

where $D$ is the area in the triangle.

(b) By Green’s Theorem, we have

$$\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \iint_D (-1 + 2y^2 - 6x^2)dxdy = \int_0^a \int_{-y}^y (-1 + 2y^2 - 6x^2) \, dy \, dx = \int_0^a -2ydy = -a^2.$$

Hence, by $-a^2 = -1$ and $a > 0$ we can solve $a = 1$.

6. (a) Let $f_1(x, y, z) = y$, $f_2(x, y, z) = x$, and $f_3(x, y, z) = 1$. We can find $g(x, y, z) = xy + z$ satisfying $\frac{\partial g}{\partial x} = f_1$, $\frac{\partial g}{\partial y} = f_2$, and $\frac{\partial g}{\partial z} = f_3$. Hence, the set of integrals of the form $\int_C (y \, dx + x \, dy + dz)$ is path independent.

Returning to the curve $C$ given in the problem, we know that it starts from point $(0, 0, 0)$ and ends at $(4\pi, 0, 6\pi)$. Therefore, $\int_C (y \, dx + x \, dy + dz) = g(4\pi, 0, 6\pi) - g(0, 0, 0) = 6\pi$.

(b) We can represent the intersection of the two surfaces in a parametric form $\mathbf{r}(t) = [x(t), y(t), z(t)]$ where

$$x(t) = 2t,$$
$$y(t) = t,$$
$$z(t) = 5t^2 - 1.$$

$C$ is the part of the above curve from $t = 0$ to $t = 1$. Hence:

$$\int_C (y \, dx - x \, dy + dz) = \int_0^1 (t \frac{dx}{dt} - 2t \frac{dy}{dt} + \frac{dz}{dt})dt = \int_0^1 (2t - 2t + 10t)dt = 5.$$