In this lecture, we will pave a stepping stone for our subsequent discussion on surface integrals by discussing a topic that is interesting in its own right: the relationship between the area of a planar region embedded in $\mathbb{R}^3$ and the area of its projection onto the xy-plane.

1 Projection of a Parallelogram

Let us start with the following problem. In $\mathbb{R}^3$, we are given a parallelogram $g$ that is in a plane $\rho$ with a normal vector $u$. Now, project $g$ onto the xy-plane, which gives us another parallelogram $g_{xy}$; see the figure below. Denote by $A$ the area of $g$, and by $A_{xy}$ the area of $g_{xy}$. We want to explore the relationship between $A$ and $A_{xy}$.

Denote by $\gamma$ the angle between the directions of $u$ and $k$ (i.e., the positive z-direction). Next, we prove a very neat result:

**Lemma 1.** $A_{xy} = A \cdot |\cos \gamma|$. 

**Proof.** If $A = 0$ (i.e., $g$ degenerates into a point), then $A_{xy}$ is trivially 0, in which case the lemma is obviously true. Next, we consider that $A \neq 0$.

Consider first $\gamma \in [0, \pi/2]$. Let $a$ and $b$ be the vectors corresponding to the two directed segments as shown in the above figure. Let $a'$ and $b'$ be the projections of $a$ and $b$ onto the xy-plane, respectively. If we write out the components of $a$ and $b$ as:

$$a = [x_1, y_1, z_1]$$
$$b = [x_2, y_2, z_2]$$

$$A_{xy} = a' \cdot b' = a' \cdot b = |a' \cdot b|$$

where $a'$ and $b'$ are the projections of $a$ and $b$ onto the xy-plane, respectively. Since $\gamma \in [0, \pi/2]$, we have:

$$|\cos \gamma| = \cos \gamma$$

Therefore, we have:

$$A_{xy} = A \cdot |\cos \gamma|$$
then we have

\[ a' = [x_1, y_1, 0] \]
\[ b' = [x_2, y_2, 0]. \]

We know that the areas of \( g \) and \( g_{xy} \) are

\[ A = |a \times b| \]
\[ A_{xy} = |a' \times b'|. \]

Define \( c = a \times b \) and \( c' = a' \times b' \). By definition of cross product, we know:

\[ c = [y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2] \]
\[ c' = [0, 0, x_1 y_2 - y_1 x_2]. \]

The directions of \( c \) and \( c' \) are shown in the above figure. Note that \( \gamma \) is also the angle between \( c \) and \( c' \).

We thus have:

\[ A = |c| \]
\[ A_{xy} = |c'|. \]

If \( c' = 0 \), then it means that \( a' \) and \( b' \) have exactly the same or opposite directions, which further implies that \( \gamma = \pi/2 \). In this case, we trivially have \( A_{xy} = 0 = A \cos \gamma \). If \( c' \neq 0 \), we have

\[ \cos \gamma = \frac{c \cdot c'}{|c||c'|} = \frac{(x_1 y_2 - y_1 x_2)^2}{|c||c'|} = \frac{|c'|^2}{|c||c'|} = \frac{|c'|}{|c|} = \frac{A_{xy}}{A} \]

which is precisely what we want to prove.

For the case where \( \gamma \in [\pi/2, \pi] \), let \( v = -u \). The angle between the directions of \( v \) and \( k \) is within \([\pi/2, \pi]\). Now we can apply the above argument with respect to the normal vector \( v \) to establish the lemma.

\[ \square \]

**Example 1.** Consider the plane \( \rho \) given by \( x + 2y + 3z = 4 \). Let \( D \) a rectangle on the xy-plane with area 1, and \( D' \) the projection of \( D \) onto \( \rho \). What is the area of \( D' \)?

**Solution.** A normal vector of \( \rho \) is \( u = [1, 2, 3] \). Let \( \gamma \) be the angle between \( u \) and \( k = [0, 0, 1] \). We know that \( \cos \gamma = \frac{u \cdot k}{|u||k|} = \frac{3}{\sqrt{14}} \). Hence, by Lemma 1, the area of \( D' \) equals \( \text{area}(D)/\cos \gamma = \sqrt{14}/3 \).

\[ \square \]

### 2 Projection of Any Planar Region

We now generalize Lemma 1. In \( \mathbb{R}^3 \), we are given an arbitrary region \( D \) that is in a plane \( \rho \) with a normal vector \( u \). Suppose that the boundary of \( D \) is a smooth curve. Now, project \( D \) onto the xy-plane, which gives us another region \( D_{xy} \); see the figure below.
Denote by \( \gamma \) the angle between the directions of \( \mathbf{u} \) and \( \mathbf{k} \). In general, we still have:

**Lemma 2.** \( \text{area}(D_{xy}) = \text{area}(D) \cdot |\cos \gamma| \).

We will not prove the lemma formally, but its key idea is easy to grasp. Imagine that we approximate \( D \) as the union of a huge number of very small disjoint parallelograms, and project all those parallelograms onto the xy-plane. The union of those parallelograms’ projections approximates \( D_{xy} \). Then, by Lemma 1, there is a \( \cos \gamma \) factor between the areas of each parallelogram and its projection, which thus gives Lemma 2.

**Example 2.** Consider the plane \( \rho \) given by \( x + 2y + 3z = 4 \). Let \( D \) be a circle on the xy-plane with radius 1, and \( D' \) the projection of \( D \) onto \( \rho \). What is the area of \( D' \)?

**Solution.** A normal vector of \( \rho \) is \( \mathbf{u} = [1, 2, 3] \). Let \( \gamma \) be the angle between \( \mathbf{u} \) and \( \mathbf{k} = [0, 0, 1] \). We know that \( \cos \gamma = \frac{\mathbf{u} \cdot \mathbf{k}}{||\mathbf{u}|| ||\mathbf{k}||} = \frac{3}{\sqrt{14}} \). Hence, by Lemma 2, the area of \( D' \) equals \( \frac{\text{area}(D)}{\cos \gamma} = \frac{\sqrt{14} \pi}{3} \). \( \square \)