1 Smooth, xy-Monotone, and Oriented Surfaces

Recall that one way to specify a surface in $\mathbb{R}^3$ is to give an equation $f(x, y, z) = 0$ over some legal ranges of $x, y, z$. We say that the surface is smooth if both of the following are satisfied:

- the gradient $\nabla f(p) = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$ changes continuously as point $p$ moves about on the surface;
- $\nabla f(p) \neq 0$.

As discussed earlier, $\nabla f(p)$ gives a normal vector of the surface at point $p$. Hence, the first bullet essentially says that this normal vector changes continuously as $p$ moves on the surface. The second bullet implies that we can always obtain a unit normal vector at $p$ as $\frac{\nabla f(p)}{|\nabla f(p)|}$.

We say that a surface is $xy$-monotone if every line perpendicular to the $xy$-plane hits the surface at no more than one point. In other words, the surface can be represented as an equation $z = g(x, y)$. For example, the sphere $x^2 + y^2 + z^2 = 1$ is not $xy$-monotone, but the hemisphere

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ z \geq 0 \end{cases}$$

is, because we can represent the hemisphere as $z = \sqrt{1 - x^2 - y^2}$.

An $xy$-monotone surface $S$ usually has two “sides”. For example, the cap-shaped surface below has two sides: facing outward and inward, respectively. To define surface integral by coordinate, we need to choose a side of the surface. Formally, we do so by choosing the directions of normal vectors. Specifically, for each point $p$ on the surface $S$, take a normal vector $\mathbf{u}$ at $p$. There are only two choices for $\mathbf{u}$, as shown in the example below. Denote by $\gamma(p)$ the angle between the direction of $\mathbf{u}$ and the positive direction of the $z$-axis. We require that either $\gamma(p) \in [0, \pi/2]$ for all $p$ on $S$, or $\gamma(p) \in [\pi/2, \pi]$ for all $p$ on $S$. In the former case, we say that we have chosen the upper side of $S$, where in the latter, we say that we have chosen the lower side. In both cases, $S$ is said to have been oriented.

![Diagram of a surface with normal vectors and an angle marked]
2 Surface Integral by Coordinates x and y

Let $S$ be an oriented xy-monotone surface described by equation $z = g(x, y)$. Let $D$ be the projection of $S$ onto the xy-plane. We say that function $h(x, y, z)$ is continuous on $S$ if $h(x, y, g(x, y))$ is continuous in $D$. Then, we define surface integral

$$\iint_S h(x, y, z) \, dxdy$$

as a short form for

$$\begin{cases} \iint_D h(x, y, g(x, y)) \, dxdy & \text{if } S \text{ is the upper side of } z = g(x, y) \\ -\iint_D h(x, y, g(x, y)) \, dxdy & \text{otherwise} \end{cases}$$

Example 1. Let $S$ be the lower side of the plane $3x + 2y + z = 6$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Calculate $\iint_S x + y + z \, dxdy$.

Solution. Let $D$ be the area in the xy-plane corresponding to $0 \leq x \leq 1$ and $0 \leq y \leq 1$. $S$ can be described by the equation $z = 6 - 3x - 2y$.

$$\iint_S x + y + z \, dxdy = -\iint_D x + y + 6 - 3x - 2y \, dxdy = -\iint_D 6 - 2x - y \, dxdy = -9/2.$$  

3 Evaluating Surface Integrals by Jacobian

Recall that a surface is inherently a 2D geometric object, even though it is embedded in $\mathbb{R}^3$. Besides using an equation $f(x, y, z) = 0$, we can also describe a surface by representing x-, y-, and z-coordinates as functions of two parameters $u$ and $v$, namely, $x(u, v)$, $y(u, v)$, $z(u, v)$. Accordingly, we can evaluate a surface integral by changing the integral variables from $x,y$ to $u,v$. However, since we are dealing with a double integral, the change of variables is more complicated than simply applying the chain rule; instead, we need to resorting to the Jacobian, as you should have learned in a prerequisite course. Next, we illustrate this using an example.

Example 2. Let $S$ be the upper side of the hemisphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. Calculate $\iint_S z^2 \, dxdy$.

Solution. Let $D$ be the projected region of $S$ onto the xy-plane, namely, $D$ is the disc $x^2 + y^2 \leq 1$. Hence:

$$\iint_S z^2 \, dxdy = \iint_D 1 - x^2 - y^2 \, dxdy.$$  

We can represent the x-, y-, and z-coordinates of each point $(x, y, z)$ on $S$ as functions of $u,v$:

$$\begin{align*} x(u, v) &= \cos u \cdot \sin v \\ y(u, v) &= \sin u \cdot \sin v \\ z(u, v) &= \cos v \end{align*}$$
with $0 \leq u \leq 2\pi$ and $0 \leq v \leq \pi/2$. The Jacobian $J$ equals:

$$J = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

$$= - \sin u \cdot \sin v \cdot \sin u - \cos u \cdot \cos v \cdot \cos u \cdot \sin v$$

$$= - \sin v \cdot \cos v.$$

Now we can change the variables $x, y$ in (2) to $u, v$ as:

$$\int\int_D 1 - x^2 - y^2 \, dxdy = \int\int_D (1 - \cos^2 u \sin^2 v - \sin^2 u \sin^2 v) \cdot |J| \, dudv$$

$$= \int\int_D \cos^2 v \cdot |J| \, dudv$$

$$= \int\int_D \cos^2 v \cdot |\sin v \cdot \cos v| \, dudv$$

$$= \int_0^{2\pi} \left( \int_0^{\pi/2} \cos^3 v \cdot \sin v \, dv \right) \, du$$

$$= \frac{1}{4} \int_0^{2\pi} du = \pi/2.$$

4 Surface Integrals on Regions Not xy-Monotone

So far our definition of surface integrals in (1) is limited to xy-monotone regions. Next, we extend the definition also to regions that are not xy-monotone. We achieve the purpose by (i) introducing a special case for vertical regions, and (ii) cutting a non-xy-monotone region into xy-monotone ones.

A Special Case. Let $S$ be a surface that is perpendicular to the xy-plane. Then, we define

$$\int\int_S h(x, y, z) \, dxdy = 0.$$

The above definition is fairly intuitive. If $S$ is perpendicular to the xy-plane, its projection $D$ onto the xy-plane is a line segment whose area is 0; see below.
**Piecewise xy-Monotone Surfaces.** Let $S$ be a surface that can be cut into a sequence of surfaces $S_1, S_2, ..., S_m$, each of which is either an oriented surface, or perpendicular to the xy-plane. We refer to $S$ as a *piecewise xy-monotone surface*. Also, suppose that function $h(x, y, z)$ is continuous on each xy-monotone $S_i (i \in [1, m])$. Then, we define

$$
\iint_S h(x, y, z) \, dx \, dy = \sum_{i=1}^m \iint_{S_i} h(x, y, z) \, dx \, dy.
$$

**Example 3.** Let $S$ be the outer side of the sphere $x^2 + y^2 + z^2 = 1$. Calculate $\iint_S z^2 \, dx \, dy$.

**Solution.** Divide $S$ into two xy-monotone surfaces $S_1$ and $S_2$, where

- $S_1$ is the upper side of $x^2 + y^2 + z^2 = 1$ with $z \geq 0$;
- $S_2$ is the lower side of $x^2 + y^2 + z^2 = 1$ with $z \leq 0$.

Thus:

$$
\iint_S h(x, y, z) \, dx \, dy = \iint_{S_1} h(x, y, z) \, dx \, dy + \iint_{S_2} h(x, y, z) \, dx \, dy.
$$

We have seen in Example 2 that $\iint_{S_1} h(x, y, z) \, dx \, dy = \pi/2$. Next, we calculate $\iint_{S_2} h(x, y, z) \, dx \, dy$.

Let $D$ be the projected region of $S_2$ onto the xy-plane, namely, $D$ is the disc $x^2 + y^2 \leq 1$. Hence:

$$
\iint_{S_2} z^2 \, dx \, dy = - \iint_{D} 1 - x^2 - y^2 \, dx \, dy. \tag{3}
$$

We can represent the x-, y-, and z-coordinates of each point $(x, y, z)$ on $S$ as functions of $u, v$:

$$
\begin{align*}
x(u, v) & = \cos u \cdot \sin v \\
y(u, v) & = \sin u \cdot \sin v \\
z(u, v) & = \cos v
\end{align*}
$$

with $0 \leq u \leq 2\pi$ and $\pi/2 \leq v \leq \pi$. The Jacobian $J$ equals:

$$
J = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = - \sin v \cdot \cos v.
$$

Now we can change the variables $x, y$ in (3) to $u, v$ as:

$$
- \iint_{D} 1 - x^2 - y^2 \, dx \, dy = - \iint_{D} \cos^2 v \cdot |J| \, du \, dv.
$$

$$
= - \iint_{D} \cos^2 v \cdot |\sin v \cdot \cos v| \, du \, dv
$$

$$
= \int_0^{2\pi} \left( \int_{\pi/2}^\pi \cos^3 v \cdot \sin v \, dv \right) \, du
$$

$$
= \frac{1}{4} \int_0^{2\pi} du = -\pi/2.
$$

Therefore, $\iint_S h(x, y, z) \, dx \, dy = \pi/2 - \pi/2 = 0$. \qed