Lecture Notes: Path Independence of Certain Line Integrals (Part 2)

Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

1 2D

Define $S$ to be the set of line integrals of the form

$$\int_C f_1 \, dx + \int_C f_2 \, dy.$$  

In the previous lecture, we learned:

**Theorem 1 (Proved Previously).** $S$ is path independent if and only if we can find a function $g(x, y)$ such that

$$f_1(x, y) = \frac{\partial g}{\partial x}, \text{ and } f_2(x, y) = \frac{\partial g}{\partial y}. \quad (1)$$

**Example 1.** Prove that the set of integrals of the form

$$\int_C y^2 \sin(x + x \cdot \cos(x)) \, dx + \int_C 2xy \sin(x) \, dy. \quad (2)$$

is path independent.

**Proof.** Let $g(x, y) = x \sin(x) \cdot y^2$. We have that $\frac{\partial g}{\partial x} = y^2 \sin(x) + x \cos(x)$ and $\frac{\partial g}{\partial y} = 2xy \sin(x)$. Hence, by Theorem 1, the set of integrals (2) is path independent. \hfill $\square$

Proving path independence by Theorem 1 demands the ability of observing $g(x, y)$. What if such $g(x, y)$ is difficult to observe (as may be the case in the previous example)? Fortunately, it is often easy to determine whether $S$ is path independent without deriving $g(x, y)$. This is shown in the following Theorem:

**Theorem 2.** Suppose that $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial x}$ are both continuous in $\mathbb{R}^2$. $S$ is path independent if and only if

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}. \quad (3)$$

**Proof. The Only-If Direction.** Suppose that $S$ is path independent. We want to prove that (3) must hold. Since $S$ is path independent, by Theorem 1, there is a function $g(x, y)$ satisfying (1). Therefore, $\frac{\partial f_1}{\partial y} = \frac{\partial^2 g}{\partial x \partial y}$ and $\frac{\partial f_2}{\partial x} = \frac{\partial^2 g}{\partial y \partial x}$. The continuity of $\frac{\partial f_1}{\partial y}$ and $\frac{\partial f_2}{\partial x}$ determines that $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$, which leads to (3).

**The If-Direction.** The proof requires the Green’s Theorem, which will be introduced later in the course. \hfill $\square$
Example 1 (Revisited). Prove that the set of integrals of the form
\[ \int_C y^2(\sin(x) + x \cdot \cos(x)) \, dx + \int_C 2xy \sin(x) \, dy. \tag{4} \]
is path independent.

Proof. Let \( f_1(x, y) = y^2(\sin(x) + x \cdot \cos(x)) \) and \( f_2(x, y) = 2xy \sin(x) \). We have that \( \frac{\partial f_1}{\partial y} = 2y(\sin(x) + x \cos(x)) \) and \( \frac{\partial f_2}{\partial x} = 2y(\sin(x) + x \cos(x)) \). Hence, by Theorem 2, the set of integrals (4) is path independent. \( \square \)

2 3D

The discussion in the previous section can be generalized to \( \mathbb{R}^3 \). Let \( f_1(x, y, z) \), \( f_2(x, y, z) \), and \( f_3(x, y, z) \) be scalar functions. Define \( S \) to be the set of all possible line integrals of the form
\[ \int_C f_1 \, dx_1 + \int_C f_2 \, dx_2 + \int_C f_3 \, dx_3. \tag{5} \]

Theorem 3. Suppose that \( \frac{\partial f_1}{\partial y} \), \( \frac{\partial f_1}{\partial z} \), \( \frac{\partial f_2}{\partial x} \), \( \frac{\partial f_2}{\partial z} \), \( \frac{\partial f_3}{\partial x} \), and \( \frac{\partial f_3}{\partial y} \) are all continuous in \( \mathbb{R}^3 \). \( S \) is path independent if and only if all the following hold:
\[
\begin{align*}
\frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} \\
\frac{\partial f_2}{\partial z} &= \frac{\partial f_3}{\partial y} \\
\frac{\partial f_3}{\partial x} &= \frac{\partial f_1}{\partial z}.
\end{align*}
\]

Proof. The only-if direction is a direct extension of the proof of Theorem 2. The if-direction requires the Stokes’s Theorem, which will be introduced later in the course. \( \square \)

Example 2. Prove that the set of integrals of the form:
\[ \int_C 2x y^2 \, dx + \int_C 2x^2 yz \, dy + \int_C x^2 y^2 \, dz \tag{6} \]
is path independent.

Proof. Let \( f_1(x, y, z) = 2x y^2 z \), \( f_2(x, y, z) = 2x^2 yz \), and \( f_3(x, y, z) = x^2 y^2 \). We have that
\[
\begin{align*}
\frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} = 4xyz \\
\frac{\partial f_2}{\partial z} &= \frac{\partial f_1}{\partial z} = 2xy^2 \\
\frac{\partial f_3}{\partial x} &= \frac{\partial f_3}{\partial y} = 2x^2 y.
\end{align*}
\]
Hence, by Theorem 3, the set of integrals (6) is path independent. \( \square \)

It is worth mentioning that Theorem 3 is closely related to a concept of curl defined as follows:
Definition 1. Let \( f(x, y, z) \) be a vector function defined as
\[
f(x, y, z) = [f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)].
\]
Then, the curl of \( f(x, y, z) \) is defined as:
\[
\text{curl } f = [h_1(x, y, z), h_2(x, y, z), h_3(x, y, z)]
\]
where
\[
h_1(x, y, z) = \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z},
\]
\[
h_2(x, y, z) = \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x},
\]
\[
h_3(x, y, z) = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}.
\]
Hence, by Theorem 3, the set of integrals (5) is path independent if and only if \( \text{curl } f = 0 \).

Example 3. Define \( f(x, y, z) = [xyz, x^2, y^2z] \). Then,
\[
\text{curl } f = [2yz - 2x, xy, 2x - xz].
\]
By Theorem 3, we know that the set of integrals of the form:
\[
\int_C xyz\, dx + \int_C x^2\, dy + \int_C y^2z\, dz
\]
is not path independent.