Exercises: Path Independence of Line Integral 2

Judge if the following line integrals are path independent. If so, calculate the integral on a curve from point \((0,0,0)\) to point \((1,1)\) in 2d, or from point \((0,0,0)\) to point \((1,1,1)\) in 3d.

**Problem 1.** \[ \int_C 2e^{x^2}(x \cos(2y)\, dx - \sin(2y)\, dy). \]

**Solution:** Let \(f_1(x,y) = 2e^{x^2}\cdot x \cos(2y)\) and \(f_2(x,y) = -2e^{x^2}\cdot \sin(2y).\) Thus, \(\frac{\partial f_1}{\partial y} = -4xe^{x^2}\sin(2y)\) and \(\frac{\partial f_2}{\partial x} = -4xe^{x^2}\sin(2y).\) Hence, the integral is path independent.

Next, we evaluate the integral. If you can observe quickly that \(g(x,y) = e^{x^2}\cos(2y)\) satisfies \(\frac{\partial g}{\partial x} = f_1\) and \(\frac{\partial g}{\partial y} = f_2,\) then you can directly give the answer \(g(1,1) - g(0,0) = e\cos(2) - 1.\)

Suppose that you cannot observe the above \(g(x,y)\) directly. Here would be another way of solving the line integral. Choose a curve \(C\) on which the integral is easy to evaluate. Let \(C\) be the concatenation of two curves: \(C_1\) from \((0,0)\) to \((1,0)\), and \(C_2\) from \((1,0)\) to \((1,1)\). We first evaluate

\[
\int_{C_1} 2e^{x^2}(x \cos(2y)\, dx - \sin(2y)\, dy) = \int_{C_1} 2e^{x^2} x \cos(2y)\, dx \\
= \int_0^1 2e^{x^2} x \cos(2 \cdot 0)\, dx \\
= \int_0^1 2e^{x^2} \, dx \\
= \int_0^1 e^{x^2} \, d(x^2) = e - 1
\]

Then evaluate

\[
\int_{C_2} 2e^{x^2}(x \cos(2y)\, dx - \sin(2y)\, dy) = -\int_{C_2} 2e^{x^2} \sin(2y)\, dy \\
= -\int_0^1 2e \sin(2y)\, dy = e \cos(2) - e
\]

Hence, \(\int_C 2e^{x^2}(x \cos(2y)\, dx - \sin(2y)\, dy)\) equals \(e - 1 + e \cos(2) - e = e \cos(2) - 1.\)

**Problem 2.** \[ \int_C (x^2y\, dx - 4xy^2\, dy + 8z^2\, x\, dz). \]

**Solutions:** Let \(f_1 = x^2y,\) \(f_2 = -4xy^2,\) and \(f_3 = 8z^2x.\) Hence, \(\frac{\partial f_1}{\partial y} = x^2\) and \(\frac{\partial f_2}{\partial x} = -4y^2.\) Since \(\frac{\partial f_1}{\partial y} \neq \frac{\partial f_2}{\partial x},\) we conclude that the integral is not path independent.

**Problem 3.** \[ \int_C (e^y\, dx + (xe^y - e^z)\, dy - ye^z\, dz). \]

**Solutions:** Let \(f_1 = e^y,\) \(f_2 = xe^y - e^z,\) and \(f_3 = -ye^z.\) Hence, \(\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} = e^y,\) \(\frac{\partial f_1}{\partial z} = \frac{\partial f_2}{\partial y} = 0,\) and \(\frac{\partial f_3}{\partial x} = \frac{\partial f_3}{\partial y} = -e^z.\) Hence, the integral is path independent.

Next, we evaluate the integral. If you can observe quickly that \(g(x,y,z) = xe^y - ye^z\) satisfies \(\frac{\partial g}{\partial x} = f_1,\) \(\frac{\partial g}{\partial y} = f_2,\) and \(\frac{\partial g}{\partial z} = f_3,\) then you can directly give the answer \(g(1,1,1) - g(0,0,0) = 0.\)

Suppose that you cannot observe the above \(g(x,y)\) directly. Choose a curve \(C\) on which the integral is easy to evaluate. Let \(C\) be the concatenation of three curves: \(C_1\) from \((0,0,0)\) to \((0,0,1),\)
\[ C_2 \text{ from } (0,0,1) \text{ to } (0,1,1), \text{ and } C_3 \text{ from } (0,1,1) \text{ to } (1,1,1). \] We first evaluate
\[
\int_{C_1} (e^y \, dx + (xe^y - e^z) \, dy - ye^z \, dz) = - \int_{C_1} ye^z \, dz \\
= - \int_0^1 0e^z \, dz = 0.
\]
Then evaluate:
\[
\int_{C_2} (e^y \, dx + (xe^y - e^z) \, dy - ye^z \, dz) = \int_{C_2} (xe^y - e^z) \, dy \\
= \int_0^1 -e \, dy = -e.
\]
Finally evaluate:
\[
\int_{C_3} (e^y \, dx + (xe^y - e^z) \, dy - ye^z \, dz) = \int_{C_3} e^y \, dx \\
= \int_0^1 e \, dx = e.
\]
Hence, \( \int_C (e^y \, dx + (xe^y - e^z) \, dy - ye^z \, dz) = 0 - e + e = 0. \)

**Problem 4.** \( \int_C (4y \, dx + (4x + z) \, dy + (y - 2z) \, dz). \)

**Solutions:** Let \( f_1 = 4y, f_2 = 4x + z, \) and \( f_3 = y - 2z. \) Hence, \( \frac{\partial f_1}{\partial y} = 4, \frac{\partial f_2}{\partial x} = 4, \frac{\partial f_1}{\partial z} = \frac{\partial f_2}{\partial y} = 0, \) and \( \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} = 1. \) Hence, the integral is path independent.

Next, we evaluate the integral. If you can observe quickly that \( g(x, y, z) = 4xy + yz - z^2 \) satisfies \( \frac{\partial g}{\partial x} = f_1, \frac{\partial g}{\partial y} = f_2, \) and \( \frac{\partial g}{\partial z} = f_3, \) then you can directly give the answer \( g(1,1,1) - g(0,0,0) = 4. \)

Suppose that you cannot observe the above \( g(x, y) \) directly. Choose a curve \( C \) on which the integral is easy to evaluate. Let \( C \) be the line segment given by \( r(t) = [x(t), y(t), z(t)] \) with \( x(t) = y(t) = z(t) = t, \) and \( t \in [0,1]. \) Then
\[
\int_C (4y \, dx + (4x + z) \, dy + (y - 2z) \, dz) = \int_0^1 (4t \frac{dx}{dt} + (4t + t) \frac{dy}{dt} + (t - 2t) \frac{dz}{dt}) \, dt \\
= \int_0^1 (4t + 5t - t) \, dt = 4.
\]