Exercises: Line Integral by Length

Problem 1. Let $C$ be the curve from point $p(0,0)$ to point $q(1,1)$ on the parabola $y = x^2$. Calculate $\int_C x \, ds$.

Solution: First, write $C$ into its parametric form: $\mathbf{r}(t) = [x(t), y(t)]$ where $x(t) = t$, and $y(t) = t^2$. Points $p$ and $q$ are given by $t = 0$ and $1$, respectively. Thus:

$$\int_C x \, ds = \int_0^1 x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \int_0^1 t \sqrt{1 + 4t^2} \, dt$$

$$= \frac{1}{12}(1 + 4t^2)^{3/2}\bigg|_0^1 = 5\sqrt{5} - \frac{1}{12}.$$  

Problem 2. Let $C$ be the line segment from point $p(1,2,3)$ to point $q(8,7,6)$. Calculate $\int_C x + z^2 \, ds$.

Solution: Vector $\mathbf{q} - \mathbf{p} = [8,7,6] - [1,2,3] = [7,5,3]$ gives the direction of the line segment. Hence, $C$ can be written into its parametric form: $\mathbf{r}(t) = [x(t), y(t), z(t)]$ where $x(t) = 1 + 7t$, $y(t) = 2 + 5t$, and $z(t) = 3 + 3t$. Points $p$ and $q$ are given by $t = 0$ and $t = 1$, respectively. Thus:

$$\int_C x + z^2 \, ds = \int_0^1 (x(t) + (z(t))^2) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$= \int_0^1 \sqrt{83} (10 + 25t + 9t^2) \, dt$$

$$= \frac{51\sqrt{83}}{2}.$$  

Problem 3. Let $C$ be the circle $x^2 + y^2 = 1$. Calculate $\int_C y \, ds$.

Solution: Note that $C$ is a closed circle. Next, we give two methods to solve the problem, which illustrate two different ways to deal with closed curves.

Method 1. Choose two arbitrary points on $C$, e.g., $p(1,0)$ and $q(-1,0)$. Break $C$ into two curves: (i) $C_1$ from $p$ counterclockwise to $q$, and (ii) $C_2$ from $q$ counterclockwise to $p$. We will calculate $\int_{C_1} y \, ds$ and $\int_{C_2} y \, ds$ separately.

Introduce the parametric form of $C$: $\mathbf{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$.
For $C_1$, $p$ and $q$ are given by $t = 0$ and $\pi$, respectively. Thus:

\[
\int_{C_1} y \, ds = \int_0^\pi \sin(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_0^\pi \sin(t) \sqrt{(-\sin(t))^2 + (\cos(t))^2} \, dt
\]

\[
= \int_0^\pi \sin(t) \, dt
\]

\[
= -\cos(t) \bigg|_0^\pi = 2
\]

For $C_2$, $q$ and $p$ are given by $t = \pi$ and $2\pi$, respectively. Thus:

\[
\int_{C_2} y \, ds = \int_{\pi}^{2\pi} \sin(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_{\pi}^{2\pi} \sin(t) \sqrt{(-\sin(t))^2 + (\cos(t))^2} \, dt
\]

\[
= \int_{\pi}^{2\pi} \sin(t) \, dt
\]

\[
= -\cos(t) \bigg|_{\pi}^{2\pi} = -2
\]

Hence:

\[
\int_C y \, ds = \int_{C_1} y \, ds + \int_{C_2} y \, ds = 0.
\]

**Method 2.** Introduce the parametric form of $C$: \(\mathbf{r}(t) = [x(t), y(t)]\) where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Pick an arbitrary point on $C$, e.g., $p(1,0)$. Let $p' = p$ (i.e., another copy of the same point). View $p$ as being given by $t = 0$, and $q$ as being given by $t = 2\pi$.

\[
\int_C y \, ds = \int_0^{2\pi} \sin(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

\[
= \int_0^{2\pi} \sin(t) \sqrt{(-\sin(t))^2 + (\cos(t))^2} \, dt
\]

\[
= \int_0^{2\pi} \sin(t) \, dt
\]

\[
= -\cos(t) \bigg|_0^{2\pi} = 0.
\]

**Problem 4.** Let $C$ be the intersection of two surfaces: sphere $x^2 + y^2 + z^2 = 3$ and plane $x = y$. Calculate $\int_C x^2 \, ds$.

**Solution:** The main difficulty of the problem is that the curve is given as the intersection of two surfaces. It is important to observe that the intersection is a closed curve. Introduce $x(t) = y(t) = \frac{\sqrt{3}}{\sqrt{2}} \cos(t)$ and $z(t) = \sqrt{3} \sin(t)$. Pick a point on $C$ by setting $t = 0$, which gives $p(\sqrt{3}/2, \sqrt{3}/2, 0)$. 

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What is the smallest $t$ that will give the same $p$? Clearly, the answer is $t = 2\pi$. Let $p' = p$, and view $p'$ as being given by $t = 2\pi$.

\[
\int_C x^2 \, ds = \int_0^{2\pi} 3 \frac{3}{2} (\cos(t))^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt
\]

\[
= \int_0^{2\pi} 3 \frac{3}{2} (\cos(t))^2 \sqrt{\left(-\frac{\sqrt{3}}{\sqrt{2}} \sin(t)\right)^2 + \left(-\frac{\sqrt{3}}{\sqrt{2}} \sin(t)\right)^2 + \left(\sqrt{3} \cos(t)\right)^2} \, dt
\]

\[
= \frac{3\sqrt{3}}{2} \int_0^{2\pi} (\cos(t))^2 \, dt
\]

\[
= \frac{3\sqrt{3}}{2} \left(\frac{t}{2} + \frac{\sin(2t)}{4}\right) \bigg|_0^{2\pi}
\]

\[
= \frac{3\sqrt{3}}{2} \pi.
\]