Exercises: Dot Product and Cross Product

Problem 1. Give the result of $\mathbf{a} \cdot \mathbf{b}$ for each of the following:

1. $\mathbf{a} = [1, 2], \mathbf{b} = [2, 5]$
2. $\mathbf{a} = [1, 2, 3], \mathbf{b} = [2, 5, -7]$

Problem 2. Give the result of $\mathbf{a} \times \mathbf{b}$ for each of the following:

1. $\mathbf{a} = [1, 2, 3], \mathbf{b} = [3, 2, 1]$
2. $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{b} = [3, 2, 1]$

Problem 3. In each of the following, you are given two vectors $\mathbf{a} \cdot \mathbf{b}$. Let $\gamma$ be the angle between the two vectors’ directions. Give the value of $\cos \gamma$.

1. $\mathbf{a} = [1, 2], \mathbf{b} = [2, 5]$
2. $\mathbf{a} = [1, 2, 3], \mathbf{b} = [3, 2, 1]$

Problem 4. This exercise explores the usage of dot product for calculation of projection lengths. Consider points $P(1, 2, 3), A(2, -1, 4), B(3, 2, 5)$. Let $\ell$ be the line passing $P$ and $A$. Now, let us project point $B$ onto $\ell$; denote by $C$ the projection. Calculate the distance between $P$ and $C$.

Problem 5. Let $\overrightarrow{P, A}, \overrightarrow{P, B},$ and $\overrightarrow{P, C}$ be directed segments that are not in the same plane. They determine a parallelepiped as shown below:

Suppose that $\overrightarrow{P, A}, \overrightarrow{P, B},$ and $\overrightarrow{P, C}$ are instantiations of vectors $\mathbf{a}, \mathbf{b},$ and $\mathbf{c},$ respectively. Prove that the volume of the parallelepiped equals $|\langle \mathbf{a} \times \mathbf{b} \rangle \cdot \mathbf{c}|$.

Problem 6. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\mathbf{p}$ to denote the corresponding vector $[x, y, z]$. Let $q$ be a point in $\mathbb{R}^3$, and $\mathbf{v}$ be a non-zero 3d vector. Denote by $\rho$ the plane passing $q$ that is perpendicular to the direction of $\mathbf{v}$. Prove that for any $p$ on $\rho$, it holds that $(\mathbf{p} - \mathbf{q}) \cdot \mathbf{v} = 0$.

Problem 7. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $\mathbf{p}$ to denote the corresponding vector $[x, y, z]$. Let $q$ be a point in $\mathbb{R}^3$, and $\mathbf{u}$ be a unit 3d vector (i.e., $|\mathbf{u}| = 1$). Denote by $\rho$ the plane passing $q$ that is perpendicular to the direction of $\mathbf{u}$. Prove that for any $p$ in $\mathbb{R}^3$, its distance to $\rho$ equals $|\langle \mathbf{p} - \mathbf{q} \rangle \cdot \mathbf{u}|$.

Problem 8. Consider the plane $x + 2y + 3z = 4$ in $\mathbb{R}^3$. Calculate the distance from point $(0, 0, 0)$ to the plane.
Problem 9. Consider the line $x + 2y = 4$ in $\mathbb{R}^2$. Calculate the distance from point $(0, 0)$ to the line.

Problem 10. Given a point $p(x, y, z)$ in $\mathbb{R}^3$, we use $p$ to denote the corresponding vector $[x, y, z]$. Let $q$ be a fixed point in $\mathbb{R}^3$, and $v$ a non-zero 3d vector. Given a real value $s$, $f(s)$ gives a point $p$ in $\mathbb{R}^3$ such that $p = q + s \cdot v$. As $s$ goes from $-\infty$ to $\infty$, what is the locus of $f(s)$?