Exercises: Eigenvalues, Eigenvectors, and Similarity Transformation

Problem 1. Find all the eigenvalues and eigenvectors of \( A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \).

Solution. Let \( \lambda \) be an eigenvalue of \( A \). To obtain all possible \( \lambda \), we solve the characteristic equation of \( A \) (let \( I \) be the 3 \( \times \) 3 identity matrix):

\[
\det(A - \lambda I) = 0 \Rightarrow
\begin{vmatrix}
-\lambda & 0 & 1 \\
0 & 1 - \lambda & 0 \\
1 & 0 & -\lambda
\end{vmatrix} = 0 \Rightarrow
(\lambda - 1)^2(\lambda + 1) = 0
\]

Hence, \( A \) has eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \).

To find all the eigenvectors of \( \lambda_1 = 1 \), we need to solve \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) from:

\[
(A - \lambda_1 I)x = 0 \Rightarrow
\begin{bmatrix}
-1 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

The set of solutions to the above equation—\( \text{EigenSpace}(\lambda_1) \)—includes all \( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) satisfying

\[
x_1 = u \\
x_2 = v \\
x_3 = u
\]

for any \( u, v \in \mathbb{R} \). Any non-zero vector in \( \text{EigenSpace}(\lambda_1) \) is an eigenvector of \( A \) corresponding to \( \lambda_1 \).

Similarly, to find all the eigenvectors of \( \lambda_2 = -1 \), we need to solve \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) from:

\[
(A - \lambda_2 I)x = 0 \Rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]
The set of solutions to the above equation—\( \text{EigenSpace}(\lambda_2) \)—includes all \[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\] satisfying

\[
\begin{align*}
x_1 &= u \\
x_2 &= 0 \\
x_3 &= -u
\end{align*}
\]

for any \( u \in \mathbb{R} \). Any non-zero vector in \( \text{EigenSpace}(\lambda_2) \) is an eigenvector of \( A \) corresponding to \( \lambda_2 \).

**Problem 2.** Let \( A \) be an \( n \times n \) square matrix. Prove: \( A \) and \( A^T \) have exactly the same eigenvalues.

**Proof.** Recall that an eigenvalue of a matrix is a root of the matrix’s characteristic equation, which equates the matrix’s characteristic polynomial to 0. It suffices to show that the characteristic polynomial of \( A \) is the same as that of \( A^T \). In other words, we want to show that \( \det(A - \lambda I) = \det(A^T - \lambda I) \). This is true because \( A - \lambda I = (A^T - \lambda I)^T \). \( \square \)

**Problem 3.** Let \( A \) be an \( n \times n \) square matrix. Prove: \( A^{-1} \) exists if and only if 0 is not an eigenvalue of \( A \).

**Proof.** **If-Direction.** The objective is to show that if 0 is not an eigenvalue of \( A \), then \( A^{-1} \) exists, namely, the rank of \( A \) is \( n \). Suppose, on the contrary, that the rank of \( A \) is less than \( n \). Consider the linear system \( Ax = 0 \) where \( x \) is an \( n \times 1 \) matrix. The hypothesis that \( \text{rank} \ A < n \) indicates that the system has infinitely many solutions. In other words, there exists a non-zero \( x \) satisfying \( Ax = 0 \). This, however, indicates that 0 is an eigenvalue of \( A \), which is a contradiction.

**Only-If Direction.** The objective is to show that if \( A^{-1} \) exists, then 0 is not an eigenvalue of \( A \). The existence of \( A^{-1} \) means that the rank of \( A \) is \( n \), which in turn indicates that \( Ax = 0 \) has a unique solution \( x = 0 \). In other words, there is no non-zero \( x' \) satisfying \( Ax' = 0x' \), namely, 0 is not an eigenvalue of \( A \). \( \square \)

**Problem 4.** Let \( A \) be an \( n \times n \) square matrix such that \( A^{-1} \) exists. Prove: if \( \lambda \) is an eigenvalue of \( A \), then \( 1/\lambda \) is an eigenvalue of \( A^{-1} \).

**Proof.** Since \( \lambda \) is an eigenvalue of \( A \), there is a non-zero \( n \times 1 \) matrix \( x \) satisfying

\[
\begin{align*}
Ax &= \lambda x \\
A^{-1}Ax &= \lambda A^{-1}x \\
x &= \lambda A^{-1}x \\
A^{-1}x &= (1/\lambda)x
\end{align*}
\]

which completes the proof. \( \square \)

**Problem 5.** Diagonalize the following matrix:

\[
A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}
\]

**Solution.** Matrix \( A \) has two eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \). Since (i) \( A \) is a \( 2 \times 2 \) matrix and (ii) it has 2 distinct eigenvalues, we can apply the diagonalization method we discussed in class.
Specifically, we obtain an arbitrary eigenvector $v_1$ of $\lambda_1$, say $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and, an arbitrary eigenvector $v_2$ of $\lambda_2$, say $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then, we form:

$$Q = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

by using $v_1$ and $v_2$ as the first and second columns, respectively. $Q$ has the inverse:

$$Q^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

We thus obtain the following diagonalization of $A$:

$$A = Q \text{diag}[3, 2] Q^{-1}.$$ 

**Problem 6.** Consider again the matrix $A$ in Problem 5. Calculate $A^t$ for any integer $t \geq 1$.

**Solution.** We already know that $A$:

$$A = Q \text{diag}[3, 2] Q^{-1}.$$ 

Hence:

$$A^t = Q \text{diag}[3^t, 2^t] Q^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 2^t \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3^t + 2^{t+1} & -3^t + 2^t \\ 2 \times 3^t - 2^{t+1} & 2 \times 3^t - 2^t \end{bmatrix}$$

**Problem 7.** Diagonalize the matrix $A$ in Problem 1.

**Solution.** Recall that all symmetric matrices are diagonalizable. $A$ is a $3 \times 3$ matrix. The key is to find three linearly independent eigenvectors.

From the solution of Problem 1, we know that $A$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. $\text{EigenSpace}(\lambda_1)$ includes all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying $x_1 = u$, $x_2 = v$, $x_3 = u$ for any $u, v \in \mathbb{R}$. The vector space $\text{EigenSpace}(\lambda_1)$ has dimension 2 with a basis $\{v_1, v_2\}$ where $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (given by $u = 1, v = 0$) and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (given by $u = 0, v = 1$).
Similarly, \( \text{EigenSpace}(\lambda_2) \) includes all \[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]
satisfying
\[
\begin{align*}
x_1 &= u \\
x_2 &= 0 \\
x_3 &= -u
\end{align*}
\]
for any \( u \in \mathbb{R} \). The vector space \( \text{EigenSpace}(\lambda_2) \) has dimension 1 with a basis \( \{ v_3 \} \) where \( v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \) (given by \( u = 1 \)).

So far, we have obtained three linearly independent eigenvectors \( v_1, v_2, v_3 \) of \( A \). We can then apply the diagonalization method exemplified in Problem 5 to diagonalize \( A \). Specifically, we form:

\[
Q = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & -1
\end{bmatrix}
\]

\( Q \) has the inverse:

\[
Q^{-1} = \begin{bmatrix}
1/2 & 0 & 1/2 \\
0 & 1 & 0 \\
1/2 & 0 & -1/2
\end{bmatrix}
\]

We thus obtain the following diagonalization of \( A \):

\[
A = Q \text{diag}[1, 1, -1] Q^{-1}.
\]