Exercises: Linear Systems and Matrix Inverse

Problem 1. Consider the following linear system:
\[
\begin{align*}
    x_1 + x_2 + x_3 + x_4 &= 1 \\
    3x_1 + x_2 + x_3 + x_4 &= a \\
    x_2 + 2x_3 + 2x_4 &= 3 \\
    5x_1 + 4x_2 + 3x_3 + 3x_4 &= a
\end{align*}
\]
Depending on the value of \( a \), when does the system have no solution, a unique solution, and infinitely many solutions?

Problem 2. Consider the following linear system:
\[
\begin{align*}
    2x_1 + x_2 + bx_3 &= 0 \\
    x_1 + x_2 + bx_3 &= 0 \\
    bx_1 + x_2 + 2x_3 &= 0
\end{align*}
\]
Depending on the value of \( b \), when does the system have no solution, a unique solution, and infinitely many solutions?

Problem 3. Use Cramer’s rule to solve the following linear system:
\[
\begin{align*}
    2x - 4y &= -24 \\
    5x + 2y &= 0
\end{align*}
\]

Problem 4. Compute the inverse of
\[
A = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & 0 & 1 \\
    0 & 1 & 0
\end{bmatrix}
\]

Problem 5. Compute the inverse of
\[
A = \begin{bmatrix}
    1 & 2 & 1 \\
    -2 & -3 & 1 \\
    5 & 9 & 1
\end{bmatrix}
\]

Problem 6. Let \( A \) be an \( n \times n \) matrix. Also, let \( I \) be the \( n \times n \) identity matrix. Prove: if \( A^3 = 0 \), then
\[
(I - A)^{-1} = I + A + A^2.
\]

Problem 7. Consider:
\[
A = \begin{bmatrix}
    2 & 1 & b \\
    1 & 1 & b \\
    b & 1 & 2
\end{bmatrix}
\]
Under what values of \( b \) does \( A^{-1} \) exist?