Exercises: Matrix Rank

Problem 1. Calculate the rank of the following matrix:

\[
\begin{bmatrix}
0 & 16 & 8 & 4 \\
2 & 4 & 8 & 16 \\
16 & 8 & 4 & 2 \\
4 & 8 & 16 & 2
\end{bmatrix}
\]

Problem 2. Calculate the rank of the following matrix:

\[
\begin{bmatrix}
4 & -6 & 0 \\
-6 & 0 & 1 \\
0 & 9 & -1 \\
0 & 1 & 4
\end{bmatrix}
\]

Problem 3. Judge whether the following vectors are linearly independent.

\[\begin{bmatrix}3, 0, 1, 2\end{bmatrix}\]
\[\begin{bmatrix}6, 1, 0, 0\end{bmatrix}\]
\[\begin{bmatrix}12, 1, 2, 4\end{bmatrix}\]
\[\begin{bmatrix}6, 0, 2, 4\end{bmatrix}\]
\[\begin{bmatrix}9, 0, 1, 2\end{bmatrix}\]

If they are not, find the largest number of linearly independent vectors among them.

Problem 4. Prove: if \(A\) is not square, then either the row vectors or the column vectors are linearly dependent.

Problem 5. Let \(S\) be an arbitrary set of vectors in \(\mathbb{R}^3\). Prove that there are at most 3 linearly independent vectors in \(S\).

Problem 6 (Hard). Prove: \(\text{rank}(AB) \leq \text{rank}A\).

Problem 7 (Very Hard). Prove: \(\text{rank}(A + B) \leq \text{rank}A + \text{rank}B\).