Exercises: Planar-Region Projection, Surface Areas, and Surface Integral by Area

Problem 1. Let $g$ be a region (bounded by a continuous curve) in the plane $x + y + z = 1$. Let $g_{xy}$ be the projection of $g$ onto the xy-plane. If we know that the area of $g$ is 1, what is the area of $g_{xy}$.

Solution: We know from the equation $x + y + z = 1$ that $N = [1, 1, 1]$ is a normal vector of the plane. Let $\gamma$ be the angle between $N$ and $k = [0, 0, 1]$. Thus:

$$\cos \gamma = \frac{N \cdot k}{|N||k|} = \frac{1}{\sqrt{3}}.$$ 

Hence, the area of $g_{xy}$ equals $1 \cdot \cos \gamma = 1/\sqrt{3}$.

Problem 2. Consider the surface $S: z = x^2 + y^2$ with $0 \leq z \leq 1$. Compute the area of $S$.

Solution. Let $D$ be the projection of the surface; note that $D$ is the disc $x^2 + y^2 \leq 1$. Introduce $f(x, y, z) = x^2 + y^2 - z$. We know that $S$ can be described by $f(x, y, z) = 0$.

$S$ is xy-monotone. Each point $(x, y)$ in $D$ uniquely corresponds to a point $p = (x, y, z(x, y))$ on $S$. Let $N$ be a normal vector of $S$ at $p$, and $\gamma$ the angle between $N$ and $k = [0, 0, 1]$. We know from the definition of surface area that the area of $S$ equals:

$$A = \iint_D \frac{1}{\cos \gamma} \, dx \, dy.$$ 

We know that $\nabla f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = [2x, 2y, -1]$ is a normal vector of $S$. Let us choose this normal vector as our $N$. Hence, we have:

$$\cos \gamma = \frac{\frac{\partial f}{\partial z}}{\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + (\frac{\partial f}{\partial z})^2}} = \frac{-1}{\sqrt{4x^2 + 4y^2 + 1}}.$$ 

Therefore:

$$A = \iint_D \sqrt{4x^2 + 4y^2 + 1} \, dx \, dy.$$ 

To evaluate the above double integral, we represent the points of $D$ using polar coordinates: $x = r \cos \theta$ and $y = r \sin \theta$, where $r \in [0, 1]$ and $\theta \in [0, 2\pi]$. This leads to:

$$A = \iint_D \sqrt{4r^2 + 1} \, dr \, d\theta = (5^{3/2} - 1)\pi/6.$$ 

Problem 3. Consider the surface $S$ in a parametric form $\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ where:

$$x(u, v) = u + v, \quad y(u, v) = u - v, \quad z(u, v) = uv.$$
with \((u, v)\) in the disc \(u^2 + v^2 \leq 1\). Compute the area of \(S\).

**Solution.** Define:

\[
\mathbf{r}_u = \left[ \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right] = [1, 1, v]
\]

\[
\mathbf{r}_v = \left[ \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right] = [1, -1, u].
\]

Define:

\[
\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [u + v, v - u, -2].
\]

Let \(R\) be the disc \(u^2 + v^2 \leq 1\). Therefore:

\[
A = \iint_R |\mathbf{N}| \, du \, dv
= \iint_R \sqrt{2u^2 + 2v^2 + 4} \, du \, dv
= (6^{3/2} - 8)\pi/3.
\]

**Problem 4.** Let \(S\) be the surface \(x + y + z = 1\) with \(x \in [0, 1], y \in [0, 1],\) and \(z \in [0, 1]\). Compute \(\iint_S x \, dA\).

**Solution.** Introduce \(f(x, y, z) = x + y + z - 1\). Hence, \(S\) can be described as \(f(x, y, z) = 0\). Take the gradient of \(f\):

\[
\nabla f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = [1, 1, 1],
\]

which points upwards. Let us orient \(S\) by taking its upper side. Let \(D\) be the projection of \(S\) onto the \(xy\)-plane. The figure below illustrates \(D\) (the shaded triangle).

\[
\iint_S x \, dA = \iint_D x \sqrt{\left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 + \left( \frac{\partial f}{\partial z} \right)^2} \, dx \, dy
= \iint_D x \sqrt{3} \, dx \, dy = \sqrt{3}/6.
\]

**Problem 5.** Let \(S\) be the surface \(\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)]\) where \(x(u, v) = u, y(u, v) = v, z(u, v) = u^3\) with \(u \in [0, 1]\) and \(v \in [-2, 2]\). Compute \(\iiint_S (1 + 9xz)^{1/2} \, dA\).
Solution. Define:

\[
\mathbf{r}_u = \left[ \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right] = [1, 0, 3u^2]
\]

\[
\mathbf{r}_v = \left[ \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right] = [0, 1, 0].
\]

Define:

\[
\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [-3u^2, 0, 1].
\]

Let \( \mathcal{R} \) be the set of \((u, v)\) with \(0 \leq u \leq 1\) and \(0 \leq v \leq 1\). Therefore:

\[
\iint_S (1 + 9xz)^{1/2} \, dA = \iint_{\mathcal{R}} (1 + 9xz)^{1/2} |\mathbf{N}| \, dudv
= \iint_{\mathcal{R}} (1 + 9u^4)^{1/2} \sqrt{9u^4 + 1} \, dudv = \frac{56}{5}.
\]

Problem 6. Define \( f(x, y, z) = [-x^2, y^2, 0] \). Let \( S \) be the surface \( \mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)] \) where \( x(u, v) = u, y(u, v) = v, z(u, v) = 3u - 2v \) with \(0 \leq u \leq 1\) and \(0 \leq v \leq 1\). Calculate \( \iint_S f \cdot \mathbf{n} \, dA \).

Solution. Let \( \mathcal{R} \) be the set of all \((u, v)\) with \(0 \leq u \leq 1\) and \(0 \leq v \leq 1\). Define:

\[
\mathbf{r}_u = \left[ \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right] = [1, 0, 3]
\]

\[
\mathbf{r}_v = \left[ \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right] = [0, 1, -2].
\]

Define:

\[
\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [-3, 2, 1].
\]

Recall (from the vector representation of surface integral by area as discussed in the class) that \( \mathbf{n} = \mathbf{N}/|\mathbf{N}| \). Also, as discussed in the class:

\[
\iint_S f \cdot \mathbf{n} \, dA = \iint_{\mathcal{R}} f \cdot \mathbf{N} \, dudv
= \iint_{\mathcal{R}} [-x^2, y^2, 0] \cdot [-3, 2, 1] \, dudv
= \iint_{\mathcal{R}} 3x^2 + 2y^2 \, dudv
= \iint_{\mathcal{R}} 3u^2 + 2v^2 \, dudv = \frac{5}{3}.
\]