Exercises: Surface Integral by Coordinate

Problem 1. Let $S$ be the upper side of the plane $x + y + z = 1$ with $x \geq 0$ and $y \geq 0$. Calculate $\iint_S z \, dxdy$.

Solution: Let $D$ be the projection of $S$ onto the xy-plane. In other words, $D$ is the shaded triangle as shown below:

![Diagram of triangle](image)

Hence:

$$\iint_S z \, dxdy = \iint_D 1 - x - y \, dxdy.$$  \hspace{1cm} (1)

$\iint_D dxdy$ is simply the area of the triangle, namely, $1/2$. Regarding the second term of (1):

$$\iint_D x \, dxdy = \int_0^1 \left( \int_0^{1-x} x \, dy \right) \, dx$$

$$= \int_0^1 x(1-x) \, dx = 1/6.$$  

By symmetry, we also have $\iint_D y \, dxdy = 1/6$. Therefore, (1) equals $1/2 - 1/6 - 1/6 = 1/6$.

Problem 2. Let $S$ be the inner side of the cube that has the origin and the point $(1,1,1)$ as the opposite corners (see below). Calculate $\iint_S (z^2 \, dxdy + xy \, dzdx)$. 

![Diagram of cube](image)
Solution: We can break $S$ into 6 oriented surfaces $S_1, S_2, ..., S_6$ as shown in the above figure. Each $S_i$ ($1 \leq i \leq 6$) corresponds to a face of the cube. Hence:

$$\int\int_S (z^2 \, dx \, dy + xy \, dz \, dx) = \sum_{i=1}^{6} \int\int_{S_i} (z^2 \, dx \, dy + xy \, dz \, dx).$$  \hspace{1cm} (2)$$

We have:

$$\sum_{i=1}^{6} \int\int_{S_i} z^2 \, dx \, dy = \int\int_{S_5} z^2 \, dx \, dy + \int\int_{S_6} z^2 \, dx \, dy$$

$$= \int\int_{S_5} 1 \, dx \, dy + \int\int_{S_6} 0 \, dx \, dy$$

$$= - \int_{0}^{1} \int_{0}^{1} dx \, dy = -1.$$  \hspace{1cm} (3)

Also:

$$\sum_{i=1}^{6} \int\int_{S_i} xy \, dz \, dx = \int\int_{S_1} xy \, dz \, dx + \int\int_{S_2} xy \, dz \, dx$$

$$= \int\int_{S_1} x \cdot 0 \, dz \, dx + \int\int_{S_2} x \, dz \, dx$$

$$= - \int_{0}^{1} \left( \int_{0}^{1} x \, dz \right) \, dx = -1/2.$$  \hspace{1cm} (4)

Therefore, (2) equals $-1 - 1/2 = -3/2$.

Problem 3. Let $S$ be the upper side of the surface $x^2 + y^2 + z^2 = 1$ with $\sqrt{2}/2 \leq z \leq \sqrt{3}/2$. Calculate $\int\int_S \frac{1}{z} \, dx \, dy$.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{s.png}
\caption{Graphical representation of the problem.}
\end{figure}

Solution 1: Let $D$ be the projection of $S$ onto the xy-plane. $D$ is the annulus $1/4 \leq x^2 + y^2 \leq 1/2$. Hence:

$$\int\int_S \frac{1}{z} \, dx \, dy = \int\int_D \frac{1}{z} \, dx \, dy.$$  \hspace{1cm} (3)
Let us represent $S$ in a parametric form $\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ where

\[
\begin{align*}
  x(u, v) &= \cos u \sin v \\
  y(u, v) &= \sin u \sin v \\
  z(u, v) &= \cos v
\end{align*}
\]

where $u \in [0, 2\pi]$ and $v \in [\pi/6, \pi/4]$. The Jacobian $J$ equals:

\[
J = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}
= -\sin u \cdot \sin v \cdot \sin u \cdot \cos v - \cos u \cdot \cos v \cdot \cos u \cdot \sin v
= -\sin v \cdot \cos v.
\]

Now we can change the variables $x, y$ in (3) to $u, v$ as:

\[
\int \int_{D_1} \frac{1}{z} \, dx \, dy = \int \int_{D_1} \frac{1}{z} \cdot |J| \, dudv
= \int \int_{D_1} \frac{1}{\cos v} \cdot \sin v \cdot \cos v \cdot |J| \, dudv
= \int_{2\pi}^{0} \left( \int_{\pi/6}^{\pi/4} \sin v \, dv \right) \, du
= (\sqrt{3} - \sqrt{2})\pi.
\]

**Solution 2:** We can also represent $S$ in another parametric form $\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ where

\[
\begin{align*}
  x(u, v) &= u \cos v \\
  y(u, v) &= u \sin v \\
  z(u, v) &= \sqrt{1 - u^2}
\end{align*}
\]

where $u \in [1/2, \sqrt{2}/2]$ and $v \in [0, 2\pi]$. The Jacobian $J$ equals:

\[
J = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}
= \cos v \cdot u \cos v - u(-\sin v) \cdot \sin v
= u.
\]

Now we can change the variables $x, y$ in (3) to $u, v$ as:

\[
\int \int_{D} \frac{1}{z} \, dx \, dy = \int \int_{D} \frac{1}{z} \cdot |J| \, dudv
= \int \int_{D} \frac{1}{\sqrt{1 - u^2}} \cdot |u| \, dudv
= \int_{2\pi}^{0} \left( \int_{\sqrt{1/2}}^{\sqrt{1/2}} \frac{\sqrt{1 - u^2}}{u} \, du \right) \, dv
= (\sqrt{3} - \sqrt{2})\pi.
\]