Hao Xu

Department of Computer Science and Engineering
The Chinese University of Hong Kong
Problem 1. Orthogonal set

Consider the following set $S$ with three column vectors:

$$S = \left\{ \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

Find all the possible $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that makes $S$ an orthogonal set.
Solution

For $S$ to be orthogonal, the vectors in $S$ must be mutually orthogonal to each other. We therefore have:

\[
\begin{bmatrix}
-\frac{\sqrt{2}}{2} \\
0 \\
\frac{\sqrt{2}}{2}
\end{bmatrix}
\cdot
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\cdot
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = 0
\]

which gives the following set of equations on variables $x$, $y$, and $z$:

\[-\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} z = 0\]

\[y = 0\]
Solution–cont.

The set of solutions \[
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
\] is:

\[
S = \left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}
\]
Problem 2. Orthogonal matrix

Consider the following matrix:

\[
A = \begin{bmatrix}
    -\frac{\sqrt{2}}{2} & 0 & x \\
    0 & 1 & y \\
    \frac{\sqrt{2}}{2} & 0 & z
\end{bmatrix}
\]

Find all the possible \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) that makes \( A \) an orthogonal matrix.

Recall that matrix \( A \) is orthogonal if and only if both conditions below are satisfied:

- All column vectors are mutually orthogonal.
- All column vectors have unit length.
In Problem 1, we have already obtained the set of \[
\begin{bmatrix}
x \
y \
z
\end{bmatrix}
\] satisfying the first bullet (the orthogonal constraint):

\[
S = \left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}
\]

To satisfy the "unit length" constraint, we need:

\[
\sqrt{x^2 + y^2 + z^2} = 1 \Rightarrow \\
 t^2 + t^2 = 1 \Rightarrow \\
 t = \frac{\sqrt{2}}{2} \text{ or } -\frac{\sqrt{2}}{2}
\]
Hence, there are only two \[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\] that can make \( A \) orthogonal:

\[
\begin{bmatrix}
\sqrt{2} \\
2 \\
0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
-\sqrt{2} \\
2 \\
0
\end{bmatrix}
\]
Problem 3. Symmetric Matrix Diagonalization

Diagonalize the following symmetric matrix:

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

into \( QBQ^{-1} \) where \( B \) is a diagonal matrix, and \( Q \) is an orthogonal matrix. You only need to give the details of \( Q \) and \( B \).

**Hint:** \( A \) has only two eigenvalues: \( \lambda_1 = 0 \) and \( \lambda_2 = 3 \).
We aim to obtain three eigenvectors of $A$, denoted as $v_1$, $v_2$, and $v_3$ respectively, that are (i) mutually orthogonal to each other and (ii) have lengths 1.

We first calculate the eigenspace of $\lambda_1$:

$$EigenSpace(\lambda_1) = \left\{ \begin{bmatrix} u \\ v \\ -u - v \end{bmatrix} \mid u, v \in \mathbb{R} \right\}$$

$EigenSpace(\lambda_1)$ has dimension 2. We will first take from the set two eigenvectors $x_1$, $x_2$ that are orthogonal to each other. But how?
Solution–cont.

We first set \( \mathbf{x}_1 \) to an arbitrary non-zero vector, e.g., \[
\begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix}
\]. Regarding \( \mathbf{x}_2 = \begin{bmatrix} u \\ v \\ -u - v \end{bmatrix} \), we ensure orthogonality between \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) by requiring their dot product to be 0:

\[
\begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix} \cdot \begin{bmatrix}
u \\
v \\
-u - v
\end{bmatrix} = 0
\]

\[
u + u + v = 0
\]

\[
2u + v = 0
\]

Any non-zero vector satisfying the above equation and in the form of \[
\begin{bmatrix}
u \\
v \\
-u - v
\end{bmatrix}
\] will be perpendicular to \( \mathbf{x}_1 \).
We can set $u$ to any value such that $x_2$ is not a zero-vector, e.g., $u = 1$

which gives $x_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Finally, normalize $x_1$ and $x_2$ to have length 1, which gives

$v_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ and

$v_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$. 


Next, take eigenvector from $EigenSpace(\lambda_2)$:

$$EigenSpace(\lambda_2) = \left\{ \begin{bmatrix} u \\ u \\ u \end{bmatrix} \mid u \in \mathbb{R} \right\}$$

This set has dimension 1. We take an arbitrary eigenvector, e.g.,

$$x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and normalizing this vector to length 1 gives

$$v_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$  

Therefore:

$$Q = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix},$$

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
Problem 4.

Suppose that an $n \times n$ matrix $A$ can be computed as $QBQ^{-1}$ where $Q$ is an $n \times n$ orthogonal matrix, and $B$ is an $n \times n$ diagonal matrix. Prove: $A$ is a symmetric matrix.