ENGG1410F Tutorial
More on Similarity Transformation

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I: Proving Non-diagonalizability

We will start by seeing an example where we want to prove that the following matrix is not diagonalizable:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$
I: Proving Non-diagonalizability

**Goal:** Prove that we won’t be able to find 3 linearly independent eigenvectors of $A$.

First, find the eigenvalues of $A$: $\lambda_1 = 1$ and $\lambda_2 = 2$.

We will prove:

- The eigenspace of $\lambda_1$ has dimension 1 namely, any two eigenvectors of $\lambda_1$ must be linearly dependent.
- The same is true for $\lambda_2$.

This will complete the proof.
I: Proving Non-diagonalizability

Let us first focus on $\lambda_1 = 1$. We want to solve the equation:

$$(A - \lambda_1 I)x = 0 \Rightarrow$$

$$
\begin{bmatrix}
-2 & 1 & 0 \\
0 & 1/2 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = 0
$$

We can see that there are two useful equations. In other words, there is only one unconstrained variable. Therefore, $eigenspace(\lambda_1)$ has dimension 1.
Next, focus on $\lambda_2 = 2$. We want to solve the equation:

$$(A - \lambda_2 I)x = 0 \Rightarrow$$

$$\begin{bmatrix}
-3 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 0$$

We can see that there are two useful equations. In other words, there is only one unconstrained variable. Therefore, $eigenspace(\lambda_2)$ has dimension 1.

We now can conclude that $A$ is not diagonalizable.
II: Transitivity of Diagonalizability

Let $A$, $B$, and $C$ be three $n \times n$ matrices for some integer $n$.

If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

This is an exercise in the last week’s exercise list.
III: Proof of Similarity

Prove:

\[ A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \]

is similar to

\[ B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}. \]

We will give two ways to do this.
III: Proof of Similarity

**Method 1: Use transitivity.**

Verify that $A$ and $B$ have the same eigenvalues: 3 and 2.

- By the way, if they do not, then immediately they are not similar.

Hence, $A$ can be diagonalized into $P^{-1} \text{diag}[3, 2] P$, and $B$ can be diagonalized into $Q^{-1} \text{diag}[3, 2] Q$.

In other words, $A$ and $B$ are both similar to $\text{diag}[3, 2]$. Therefore, $A$ and $B$ are similar to each other.
III: Proof of Similarity

**Method 2:** Finding an explicit form.

We will try to find an invertible matrix $P = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $A = PBP^{-1}$. Equivalently, we want to have $AP = PB$, that is:

$$
\begin{bmatrix}
1 & -1 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
x & y \\
z & w
\end{bmatrix}
\begin{bmatrix}
3 & 1 \\
0 & 2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
x - z & y - w \\
2x + 4z & 2y + 4w
\end{bmatrix}
= 
\begin{bmatrix}
3x & x + 2y \\
3z & z + 2w
\end{bmatrix}$$
Method 2: Finding an explicit form.

This gives the following equation set:

\[\begin{align*}
x - z &= 3x \\
y - w &= x + 2y \\
2x + 4z &= 3z \\
2y + 4w &= z + 2w
\end{align*}\]

You can verify that the set of solutions \(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\) is

\(\left\{ \begin{bmatrix} -u/2 \\ u/2 - v \\ u \\ v \end{bmatrix} \mid u \in \mathbb{R}, v \in \mathbb{R} \right\}\).
III: Proof of Similarity

**Method 2:** Finding an explicit form.

Let us try $u = 2$, $v = 0$. This gives $P = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$.

Since $\det(P) \neq 0$, we know that $P$ is invertible. We can now conclude that $A$ is similar to $B$. 