Problem 1. Matrix Diagonalization

Diagonalize the following matrix:

\[ A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \]
Solution

The $2 \times 2$ matrix $A$ has two distinct eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$, which means it is diagonalizable.

We then obtain an arbitrary eigenvector $v_1$ of $\lambda_1$ and also an arbitrary eigenvector $v_2$ of $\lambda_2$, say

$$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Next, apply the diagonalization method we discussed in class, form:

$$Q = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

by using $v_1$ and $v_2$ as the first and second column respectively.
Solution–cont.

$Q$ has the inverse

$$Q^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$$

We thus obtain the following diagonalization of $A$:

$$A = Q \ diag[-1, 5] \ Q^{-1}$$
Problem 2. Matrix Power

Consider again the matrix \( A \) in Problem 1, i.e.,

\[
A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}
\]

Calculate \( A^t \) for any integer \( t \geq 1 \).
We already know that

\[ A = Q \ diag[-1, 5] \ Q^{-1} \]

Hence,

\[ A^t = Q \ diag[(-1)^t, 5^t] \ Q^{-1} \]

\[
= \begin{bmatrix}
1 & 1 \\
-1 & 2
\end{bmatrix}
\begin{bmatrix}
(-1)^t & 0 \\
0 & 5^t
\end{bmatrix}
\begin{bmatrix}
2/3 & -1/3 \\
1/3 & 1/3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(5^t + 2 \times (-1)^t)/3 & (5^t + (-1)^{t+1})/3 \\
(2 \times 5^t + 2 \times (-1)^{t+1})/3 & (2 \times 5^t + (-1)^{t+2})/3
\end{bmatrix}
\]
Problem 3. Matrix Diagonalization

Diagonalize the following matrix:

\[ A = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \]
**Solution**

**A** has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. $EigenSpace(\lambda_1)$ includes all $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ satisfying $x_1 = u + v$, $x_2 = u$, $x_3 = v$ for any $u, v \in \mathbb{R}$.

The vector space $EigenSpace(\lambda_1)$ has dimension 2 with a basis $\{v_1, v_2\}$ where $v_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ (given by $u = 1, v = 0$) and $v_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ (given by $u = 0, v = 1$).

Similarly, $EigenSpace(\lambda_2)$ includes all $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ satisfying $x_1 = x_2 = -3u$ and $x_3 = u$ for any $u \in \mathbb{R}$.

The vector space $EigenSpace(\lambda_2)$ has dimension 1 with a basis $\{v_3\}$ where $v_3 = \begin{bmatrix} -3 & -3 & 1 \end{bmatrix}^T$ (given by $u = 1$).
So far, we have obtained three linearly independent eigenvectors $v_1, v_2, v_3$ of $A$. We then construct

$$Q = \begin{bmatrix}
1 & 1 & -3 \\
1 & 0 & -3 \\
0 & 1 & 1
\end{bmatrix}$$

and $Q$ has the inverse

$$Q^{-1} = \begin{bmatrix}
-3 & 4 & 3 \\
1 & -1 & 0 \\
-1 & 1 & 1
\end{bmatrix}$$

We thus obtain the following diagonalization of $A$:

$$A = Q \, diag[1, 1, 2] \, Q^{-1}$$
Problem 4. Matrix Similarity

Suppose that matrices $A$ and $B$ are similar to each other, namely, there exists $P$ such that $A = P^{-1}BP$.

Prove: if $x$ is an eigenvector of $A$ under eigenvalue $\lambda$, then $Px$ is an eigenvector of $B$ under eigenvalue $\lambda$. 
Problem 5. Matrix Trace

**Definition.** The trace of an \( n \times n \) square matrix \( A \), denoted by \( tr(A) \), is defined to be the sum of the elements on the main diagonal of \( A \), i.e.,
\[
tr(A) = \sum_{i=1}^{n} a_{ii}.
\]

For example, if
\[
A = \begin{bmatrix}
4 & -3 & -3 \\
3 & -2 & -3 \\
-1 & 1 & 2
\end{bmatrix}
\]
then \( tr(A) = 4 + (-2) + 2 = 4 \).

Prove: \( tr(AB) = tr(BA) \), where \( A \) is an \( m \times n \) matrix and \( B \) is an \( n \times m \) matrix.
Solution

**Proof.** Denote by \( a_{ij} \) the element of \( A \) at \( i \)-th row and \( j \)-th column, \( b_{ji} \) the element of \( B \) at \( j \)-th row and \( i \)-th column, where \( i = 1, 2, \cdots, m \) and \( j = 1, 2, \cdots, n \). Then

\[
(AB)_{ii} = a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni} = \sum_{j=1}^{n} a_{ij}b_{ji}
\]

Similarly,

\[
(BA)_{jj} = b_{j1}a_{1j} + b_{j2}a_{2j} + \cdots + b_{jm}a_{mj} = \sum_{i=1}^{m} b_{ji}a_{ij}
\]

Hence

\[
tr(AB) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji}a_{ij} = tr(BA)
\]

\( \square \)
Problem 6. Traces & Eigenvalues & Determinants

Suppose $A$ is an $n \times n$ diagonalizable matrix, namely, there exists $Q$ such that $A = QBQ^{-1}$, and $B$ is a diagonal matrix. Denote by $\lambda_1, \lambda_2, \cdots, \lambda_n$ the $n$ eigenvalues of $A$.

Prove: (1) $tr(A) = \sum_{i=1}^{n} \lambda_i$, (2) $det(A) = \prod_{i=1}^{n} \lambda_i$. 
Solution

Proof.

(1)

\[ tr(A) = tr(QBQ^{-1}) \]
\[ = tr(BQ^{-1}Q) \]
\[ = tr(B) \]
\[ = \sum_{i=1}^{n} \lambda_i \]

Where the second equality used the fact that \( tr(AB) = tr(BA) \) and the last equality used the facts (i) \( A \) and \( B \) have exactly the same eigenvalues due to their similarity, and (ii) the eigenvalues of a diagonal matrix are simply its diagonal elements.
(2)

\[
\text{det}(A) = \text{det}(QBQ^{-1}) \\
= \text{det}(Q) \cdot \text{det}(B) \cdot \text{det}(Q^{-1}) \\
= \text{det}(B) \cdot \text{det}(Q) \cdot \text{det}(Q^{-1}) \\
= \text{det}(B) \cdot \text{det}(QQ^{-1}) \\
= \text{det}(B) \\
= \prod_{i=1}^{n} \lambda_i
\]

Where the last equality used the facts (i) \( A \) and \( B \) have exactly the same eigenvalues due to their similarity, and (ii) the eigenvalues of a diagonal matrix are simply its diagonal elements.
In fact, the conclusion of this problem is true in general, regardless of whether $A$ is diagonalizable.

For any $n \times n$ square matrix $A$, if its $n$ eigenvalues are $\lambda_1, \lambda_2, \cdots, \lambda_n$, then $tr(A) = \sum_{i=1}^{n} \lambda_i$ and $det(A) = \prod_{i=1}^{n} \lambda_i$.

The proof is not difficult but a little tedious, students who are interested may refer to the proof at the following link: