ENGG1410-F Tutorial 5

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Problem I: Eigenvalues and eigenvectors of a Diagonal Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Calculate the eigenvalues of A.

Solution

Its characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow$$
$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

Hence, the eigenvalues are 1, 2, and 3.

Remark: What patterns can you observe about the eigenvalues of a diagonal matrix?

Problem II: Eigenvalues and eigenvectors of a matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Do the following:

- Obtain all the eigenvalues A.
- One of the eigenvalues is 1. Obtain all the eigenvectors of A corresponding to that eigenvalue.

Solution

Its characteristic equation is:

$$det(A - \lambda I) = 0 \Rightarrow$$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\text{(Expansion by 2nd col)}$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(1-\lambda)\left[(1-\lambda)^2 - 1\right] = 0 \Rightarrow$$

$$(1-\lambda)(1-\lambda-1)(1-\lambda+1) = 0 \Rightarrow$$

$$(1-\lambda)(\lambda)(2-\lambda) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 2$$

Solution Cont. - Calculate Eigenvectors

When $\lambda_1 = 1$

$$(A - \lambda_1 I)x = 0 \Rightarrow$$

$$\begin{bmatrix} 1-1 & 0 & 1 \\ 0 & 1-1 & 0 \\ 1 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

Any non-zero vectors satisfying

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ u \\ 0 \end{bmatrix}$$

(with u begin an arbitrary real value) is an eigenvector of A corresponding to λ_1.

Problem III - Problem 4 in the Exercise list

Let A be an $n \times n$ square matrix such that A^{-1} exists.
Prove: if λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.

Problem IV - Geometric Interpretation of Eigenvalue/Eigenvectors

Suppose that we map each point (x, y) into its "image" $(x, -y)$, i.e., "mirroring" the original point by the x-axis. This mapping corresponds to the following linear transformation:

$$\begin{bmatrix} x \\ -y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Problem IV - Geometric Interpretation of Eigenvalue/Eigenvectors

Now consider an eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$ of A corresponding to some eigenvalue λ. It must hold that

$$\lambda \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

Combining the above with $\begin{bmatrix} x \\ -y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$ shows that $\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$.

In other words, (x, y), $(x, -y)$, and the origin must all be on the same line! Where can (x, y) be? Answer: on the x- or y-axis!

As shown next, this is precisely the geometric interpretation of eigenvectors.

Problem IV - Geometric Interpretation of Eigenvalue/Eigenvectors

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

has eigenvalues: $\lambda_1 = 1, \lambda_2 = -1$.

For eigenvalue 1, the set of eigenvectors of A is $\left\{ \begin{bmatrix} u \\ 0 \end{bmatrix} \mid u \in \mathbb{R}, u \neq 0 \right\}$.

For eigenvalue -1, the set of eigenvectors of A is $\left\{ \begin{bmatrix} 0 \\ u \end{bmatrix} \mid u \in \mathbb{R}, u \neq 0 \right\}$.

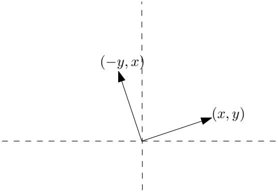
In general, let A be a square matrix. Given a point p, let q be the image of p under the linear transformation implied by A. If p is an eigenvector, then p, q, and the origin are all on the same line.

Problem IV - Geometric Interpretation of Eigenvalue/Eigenvectors

Consider now another matrix

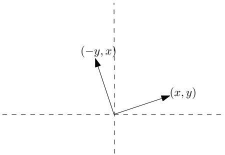
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The linear transformation of A rotates a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ by 90 degrees counterclockwise into $\begin{bmatrix} -y \\ x \end{bmatrix}$:



Problem IV - Geometric Interpretation of Eigenvalue/Eigenvectors

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



You won't be able to find p, q satisfying

> In general, let A be a square matrix. Given a point p, let q be the image of p under the linear transformation implied by A. If p is an eigenvector, then p, q, and the origin are all on the same line.

Why this oddity? Answer: A has no real-valued eigenvalues! In other words, all its eigenvalues are complex numbers. We will not be concerned with such matrices in this course.

Problem V - Problem 6 in the Exercise list

Suppose that $\lambda_1$and λ_2 are two distinct eigenvalues of matrix A. Furthermore, suppose that x_1 is an eigenvector of A under λ_1, and that x_2 is an eigenvector of A under λ_2.

Prove: there does not exist any real number c such that $cx_1 = x_2$.

Problem VI - Problem 7 in the Exercise list

Suppose that λ_1 and λ_2 are two distinct eigenvalues of matrix A. Furthermore, suppose that x_1 is an eigenvector of A under λ_1, and that x_2 is an eigenvector of A under λ_2.

Prove: $x_1 + x_2$ is not an eigenvector of A.