ENGG1410-F Tutorial:
A Closer Look at
Linear Systems with Infinite Solutions

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We learned about linear transformations. Today we will see an important application of this concept: finding all solutions to a linear system when there are infinitely many.
Let us warm up by discussing the projection of a set $V$ of vectors. Take any $V$, e.g.:

$\begin{align*}
&[3,0,1,2] \\
&[6,1,0,0] \\
&[12,1,2,4] \\
&[6,0,2,4]
\end{align*}$

The projection of $V$ onto the, say, 2nd and 3rd components is the following set $V'$ of vectors:

$\begin{align*}
&[0,1] \\
&[1,0] \\
&[1,2] \\
&[0,2]
\end{align*}$

Can you give a very short proof of the following claim: the dimension of $V$ is at least that of $V'$. 
Let us warm up by discussing the projection of a set $V$ of vectors. Take any $V$, e.g.:

\[
\begin{align*}
[3, 0, 1, 2] \\
[6, 1, 0, 0] \\
[12, 1, 2, 4] \\
[6, 0, 2, 4]
\end{align*}
\]

The projection of $V$ onto the, say, 2nd and 3rd components is the following set $V'$ of vectors:

\[
\begin{align*}
[0, 1] \\
[1, 0] \\
[1, 2] \\
[0, 2]
\end{align*}
\]

Can you give a very short proof of the following claim: the dimension of $V$ is at least that of $V'$.

**Proof:** The rank of a matrix is at least the rank of any sub-matrix. □
In general, let $V$ be any (perhaps infinite) set of vectors. By taking the same components of the vectors in $V$, we get a projection of $V$, which is a set $V'$ of vectors.

The dimension of $V$ is at least the dimension of $V'$

We leave the simple proof to you (this is actually a problem in an exercise list on the course homepage).
Now we cut into our main topic: linear system with infinitely many solutions. Consider the following system:

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 2 \\
0 & 1 & 1 & 0 & 2 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}.
\]

**Remark:** This is another problem in the same exercise list.
The system can be transformed into:

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

We can derive the set \( V \) of all the solutions \( \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} \) as follows.

First set \( x_4, x_5 \) to any real numbers (i.e., they are unconstrained). Then solve \( x_1, x_2, x_3 \) as:

\[
x_1 = - (x_4 + x_5)
\]

\[
x_2 = - x_5
\]

\[
x_3 = - x_5
\]

Now we ask the question: what is the dimension of \( V \)? Next, we show that the answer is 2, i.e., the number of variables minus the rank of the coefficient matrix!
The system can be transformed into:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}.
$$

We can derive the set $V$ of all the solutions as follows.

**First** set $x_4, x_5$ to any real numbers (i.e., they are unconstrained). **Then** solve $x_1, x_2, x_3$ as:

\begin{align*}
    x_1 &= -(x_4 + x_5) \\
    x_2 &= -x_5 \\
    x_3 &= -x_5.
\end{align*}

Now we ask the question: what is the dimension of $V$?
The system can be transformed into:

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}.
\]

We can derive the set \( V \) of all the solutions \( \begin{bmatrix}
x_1 \\
x_2 \\
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\end{bmatrix} \) as follows.

First set \( x_4, x_5 \) to any real numbers (i.e., they are unconstrained). Then solve \( x_1, x_2, x_3 \) as:

\[
\begin{align*}
    x_1 &= -(x_4 + x_5) \\
    x_2 &= -x_5 \\
    x_3 &= -x_5.
\end{align*}
\]

Now we ask the question: what is the dimension of \( V \)?

Next, we show that the answer is 2, i.e., the number of variables minus the rank of the coefficient matrix!
Denote by $V'$ the set of all vectors $\begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$.

Clearly, $V'$ has dimension 2 (remember: $x_4, x_5$ are unconstrained).

$$
\begin{align*}
x_1 &= -(x_4 + x_5) \\
x_2 &= -x_5 \\
x_3 &= -x_5 \\
x_4 &= x_4 \\
x_5 &= x_5
\end{align*}
$$

That is, $V$ can be obtained from $V'$ through a linear transformation!

We know from the lecture that linear transformations do not increase the dimension! Therefore, the dimension of $V$ is at most the dimension of $V'$. In other words, the dimension of $V$ is at most 2.
$V'$: the set of all vectors $\begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$.

\[
\begin{align*}
x_1 &= -(x_4 + x_5) \\
x_2 &= -x_5 \\
x_3 &= -x_5 \\
x_4 &= x_4 \\
x_5 &= x_5
\end{align*}
\]

On the other hand, note that $V'$ is the projection of $V$ onto the 4-th and 5-th components. From our earlier discussion, we know that the dimension of $V$ is at least the dimension of $V'$. In other words, the dimension of $V$ is at least 2.

We now conclude that the dimension of $V$ is precisely 2.