Problem 1

Suppose the following linear system $Ax = b$ has at least one non-zero solution (in which at least one of $x_1$, $x_2$ and $x_3$ is non-zero). Calculate the possible values of $\lambda$.

\[
\begin{aligned}
(1 + \lambda)x_1 + x_2 + x_3 &= 0 \\
x_1 + (1 + \lambda)x_2 + x_3 &= 0 \\
x_1 + x_2 + (1 + \lambda)x_3 &= 0
\end{aligned}
\]

**Note:** regardless of $\lambda$, $x_1 = x_2 = x_3 = 0$ is always a solution.
Solution

As mentioned, \( x_1 = x_2 = x_3 = 0 \) is always a solution. If the system has at least one non-zero solution, it means that the system has more than one solution.

According to the consistency criterion theorem, \( \det(A) = 0 \).

\[
\begin{vmatrix}
1 + \lambda & 1 & 1 \\
1 & 1 + \lambda & 1 \\
1 & 1 & 1 + \lambda
\end{vmatrix} = 0
\]

Solving the equation, we know that \( \lambda = 0 \) or \( \lambda = -3 \).
Problem 2

Calculate the determinant of the following matrix.

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 2 & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
2018 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 2019
\end{bmatrix}
\]
Solution

Let $A$ be an $n \times n$ matrix. Denote by $r_i (i = 1, 2, ..., n)$ the $i$-th row vectors of $A$.

Recall that the determinant of $A$:

- should be multiplied by 1 after switching two rows of $A$.
- should be multiplied by $c$ after multiplying all numbers of a row of $A$ by the same non-zero value $c$.
- has no change after updating row $r_i$ to $r_i + r_j$. 
Transform the given matrix into the following diagonal matrix by only using the operation of switching two columns:

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 2018 & 0 \\
0 & 0 & \cdots & 0 & 0 & 2019
\end{bmatrix}
\]

You can verify that there is an odd number of such operations (think: why). The answer is \(-2019!\) (factorial of 2019).