Problem 1. Gauss Elimination

Consider the following linear system:

\[
\begin{align*}
2y + z &= -8 \\
x - 2y - 3z &= 0 \\
-x + y + 2z &= 3
\end{align*}
\]

Solve it with Gauss Elimination.
Solution

We first obtain the augmented matrix:

\[
\begin{bmatrix}
0 & 2 & 1 & -8 \\
1 & -2 & -3 & 0 \\
-1 & 1 & 2 & 3
\end{bmatrix}
\]
Next, we convert the matrix into row echelon form:

\[
\begin{bmatrix}
0 & 2 & 1 & -8 \\
1 & -2 & -3 & 0 \\
-1 & 1 & 2 & 3
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
-1 & 1 & 2 & 3 \\
1 & -2 & -3 & 0 \\
0 & 2 & 1 & -8
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
-1 & 1 & 2 & 3 \\
0 & -1 & -1 & 3 \\
0 & 2 & 1 & -8
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
-1 & 1 & 2 & 3 \\
0 & -1 & -1 & 3 \\
0 & 0 & -1 & -2
\end{bmatrix}
\]
Now apply back substitution to obtain the solution of $x, y, z$. Specifically,

\[-z = -2 \implies z = 2\]
\[-y - z = 3 \implies y = -5\]
\[-x + y + 2z = 3 \implies x = -4\]

Therefore, the solution of the linear system is $x = -4, y = -5, z = 2$. 
Problem 2. Rank calculation

Calculate the rank of the following matrix:

\[
\begin{bmatrix}
0 & 16 & 8 & 4 \\
2 & 4 & 8 & 16 \\
16 & 8 & 4 & 2 \\
4 & 8 & 16 & 2
\end{bmatrix}
\]
Problem 3. An Important Property of Ranks

Consider the following $3 \times 5$ matrix:

$$A = \begin{bmatrix}
1 & 2 & 3 & 5 & 7 \\
1 & \sqrt{2} & \sqrt{3} & \sqrt{5} & \sqrt{7} \\
1 & 2^{1/3} & 3^{1/3} & 5^{1/3} & 7^{1/3}
\end{bmatrix}$$

Prove: there must be a column vector that is a linear combination of the other column vectors.
**Solution**

**Proof.** Denote by \( c_i \) \( (i = 1, 2, \cdots, 5) \) the \( i \)-th column vector of \( A \), since \( A \) is a \( 3 \times 5 \) matrix, we know that

\[
\text{rank} A = \text{rank} A^T \leq 3
\]

which implies that the column vectors of \( A \) are linearly dependent. In other words, there exist real values \( \alpha_1, \cdots, \alpha_5 \) such that

- they are not all zero;
- they satisfy \( \sum_{i=1}^{5} \alpha_i c_i = 0 \).

Suppose \( \alpha_k \neq 0 \) for some \( k \), then we have:

\[
c_k = - \sum_{i=1, i \neq k}^{5} \frac{\alpha_i}{\alpha_k} c_i
\]

That said, \( c_k \) is a linear combination of the other column vectors. \( \square \)
In fact, the above conclusion can be generalized, i.e.:

for an $m \times n$ matrix $A$, if $m < n$, then there must be a column vector of $A$ that is a linear combination of the other column vectors.

The proof is similar and left to you as an exercise.
Problem 4. Rank calculation

Consider a plane \( z = 2x + 3y \) in 3-dimensional space, suppose there are \( m \) points on this plane, and point \( i \) has the coordinates \((x_i, y_i, z_i)\), where \( i = 1, \cdots, m \). Let

\[
A = \begin{bmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  \vdots & \vdots & \vdots \\
  x_m & y_m & z_m
\end{bmatrix}
\]

Prove: \( \text{rank}A \leq 2 \).
**Solution**

**Proof.** Perform *elementary column operations* on $A$:

$$A = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_m & y_m & z_m \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 2x_1 + 3y_1 \\ x_2 & y_2 & 2x_2 + 3y_2 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 2x_m + 3y_m \end{bmatrix}$$

$$\implies \begin{bmatrix} x_1 & y_1 & 3y_1 \\ x_2 & y_2 & 3y_2 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 3y_m \end{bmatrix} \implies \begin{bmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 0 \end{bmatrix}$$

Hence, $\text{rank } A = \text{rank } A^T \leq 2$. □
Problem 5. Determinant calculation

Calculate the determinant of the following matrix:

\[
\begin{bmatrix}
0 & 4 & -6 \\
4 & 0 & 10 \\
-6 & 10 & 0
\end{bmatrix}
\]
Problem 6. Rank Properties

Prove: \( \text{rank}(AB) \leq \text{rank}A \).
Recall:

- Elementary row operations on a matrix do not change its rank.
- Perform an elementary row operation on a matrix $A$ is equivalent to left-multiplying $A$ by a row elementary matrix.
- The rank of a matrix of row echelon form is the number of its non-zero rows.
Proof. Denote by $A'$ the row echelon form of $A$, $E_i$ a row elementary matrix, and suppose $A'$ is obtained from $A$ by performing $z$ elementary row operations, i.e.,

$$A' = (\prod_{i=1}^{z} E_i) A = EA$$

Let $\text{rank} A = \text{rank} A' = r$, i.e., the first $r$ rows of $A'$ are non-zero, whereas the remaining rows are all zero vectors.

Suppose $A'$ is an $m \times n$ matrix, $B$ is an $n \times p$ matrix, denote the row vectors of $A'$ as $r_1, \cdots, r_m$ in top-down order and the column vectors of $B$ as $c_1, \cdots, c_p$ in left-to-right order.
Then, we have

\[ A'B = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix} \]

\[
= \begin{bmatrix}
  r_1 \cdot c_1 & r_1 \cdot c_2 & \cdots & r_1 \cdot c_p \\
  \vdots & \vdots & \ddots & \vdots \\
  r_r \cdot c_1 & r_r \cdot c_2 & \cdots & r_r \cdot c_p \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{bmatrix}
\]
Therefore,

\[ \text{rank}(AB) = \text{rank}(E(AB)) = \text{rank}((EA)B) = \text{rank}(A'B) \leq r = \text{rank}A \]

where the first inequality used the fact that performing the elementary row operations indicated by \( E \) do not change the rank of \( AB \). \( \square \)
Problem 7. Rank Properties

Let $A$ be a $m \times n$ matrix, $B$ be a $p \times q$ matrix obtained by extracting $p$ rows and $q$ columns of $A$, i.e., $B$ is a submatrix of $A$.

Prove: $\text{rank}(B) \leq \text{rank}(A)$. 
Solution

**Proof.** Denote by $r_i$ the $i$-th row vector of $A$, and $r'_j$ the $j$-th row vector of $B$, where $i = 1, \cdots, m$ and $j = 1, \cdots, p$. Assume $	ext{rank} B = r$, then there must be $r$ row vectors of $B$ that are linearly independent, let them be $r'_{x_1}, r'_{x_2}, \cdots, r'_{x_r}$, and the corresponding row vectors of $A$ are $r_{y_1}, r_{y_2}, \cdots, r_{y_r}$, where $x_k \in [1, p], y_k \in [1, m], k \in [1, r]$ and $x_k, y_k, k$ are all integers. Note that $r_{y_k}$ is an expansion of $r'_{x_k}$ for each $k$.

Then we have

$$\sum_{k=1}^{r} \alpha'_k r'_{x_k} = 0 \iff \alpha'_1 = \alpha'_2 = \cdots = \alpha'_k = 0.$$ (1)
Hence we must have
\[ \sum_{k=1}^{r} \alpha_k r_{y_k} = 0 \text{ iff } \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0. \]  \hspace{1cm} (2)

Otherwise, set \( \alpha'_k = \alpha_k \) for each \( k \) will violate (1).

(2) implies \( \text{rank} A \geq r = \text{rank} B \). \ Нечето