ENGG1410-F
Tutorial 1

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Production Problem

- In a production process, let $N$ mean “no trouble” and $T$ “trouble.” Let the transition probabilities from one day to the next be 0.8 for $N \rightarrow N$, hence 0.2 for $N \rightarrow T$, and 0.5 for $T \rightarrow N$, hence 0.5 for $T \rightarrow T$.

- If on the first day there is no trouble($N$), what is the probability of $N$ on the second today? How about the 10th day?

From Page 271, Problem 26 of the textbook
The state transition process can be modeled as the following graph:

It can also be represented as a Matrix $A = [a_{ij}]$. Call $N$ as "state 1", and $T$ as "state 2". $a_{ij}$ is the probability of moving from state $i$ to state $j$.

$$A = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}$$

The probabilities of $N$ and $T$ on the $i$-th day form a column vector $x_i$. Initially,

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
• You can verify that the probabilities of \( N \) and \( T \) on the second day can be derived as
\[
x_2 = A^T x_1 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}
\]

• The probabilities of \( N \) and \( T \) on the 10-th day can be computed as
\[
x_{10} = (A^T)^9 \cdot x_1
\]

• A fast way to compute \((A^T)^9\) is as follows
  • \((A^T)^2 = A^T A^T\)
  • \((A^T)^4 = (A^T)^2 (A^T)^2\)
  • \((A^T)^8 = (A^T)^4 (A^T)^4\)
  • \((A^T)^9 = (A^T)^8 A^T\)
Concert subscription

• In a community of 10,000 adults, there are some subscribers to a concert series and non-subscribers.

• Subscribers tend to renew their subscription in the next year with probability 0.9.

• Persons presently not subscribing will subscribe for the next year with probability 0.02.

• If the present number of subscribers is 2000, how many subscribes there would be in the 10-th year in expectation?

From Page 271, Problem 28 of the textbook
• Key observation: analyze each person separately.

• Model the state transition of a person as a graph:

• It can also be represented as a Matrix $A = [a_{ij}]$:

\[
A = \begin{bmatrix}
0.9 & 0.1 \\
0.02 & 0.98
\end{bmatrix}
\]
• A subscriber has an initial vector:
  \[ x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

• Similarly, a non-subscriber has an initial vector
  \[ x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
• For each person, her/his probability vector in the 10-th year can be obtained from
  \[ x_{10} = (A^T)^9 \cdot x_1 \]
• Let \( a \) be the number at the first row of:
  \[ (A^T)^9 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
  • Note that \( a \) is the probability that a current subscriber remains as a subscriber in the 10-th year.
• Let \( b \) be the number at the first row of:
  \[ (A^T)^9 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
  • Note that \( b \) is the probability that a current non-subscriber is a subscriber in the 10-th year.
• The expected number of subscribers in the 10-th year is
  \[ 2000 \cdot a + (10000 - 2000) \cdot b \]
Solutions to a linear system – a geometric perspective

• Given a linear system with \( m \) equations about \( n \) variables.
• For simplicity, consider \( m = n = 2 \), i.e., there are two equations and two unknown variables \( x \) and \( y \).
• If we interpret \( x \) and \( y \) as coordinates in the plane, then each equation represents a straight line.
• \((x, \ y)\) is a solution if and only if the point \( p = (x, \ y) \) lies on both lines.

From Page 273 of the textbook
Three kinds of solutions to a linear system

• For instance.

\[ x + y = 1 \]
\[ 2x - y = 0 \]
Case 1. Unique Solution

\[ x + y = 1 \]
\[ 2x + 2y = 2 \]
Case 2. Infinite Solutions

\[ x + y = 1 \]
\[ x + y = 0 \]
Case 3. No Solution
Row Elementary Matrix

- Row elementary operations can be accomplished by matrix multiplications.
- Suppose that $A$ is an $m \times n$ matrix. If $M$ is obtained from $A$ by an elementary row operation, then there is always an $m \times m$ matrix $E$ such that,
  \[ M = EA \]
- Such an $E$ is called an elementary matrix.

From Page 281, Problem 24 of the textbook
Row Elementary Matrix

• For example,
  
  \[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \]

  \[ E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \]

• \( E_1 A \leftrightarrow \) Switch the first two rows.
• \( E_2 A \leftrightarrow \) Multiply the second row of \( A \) by 2.
• \( E_3 A \leftrightarrow \) Add the 2\(^{nd}\) row into the 3\(^{rd}\) row.
Row Elementary Matrix

• Suppose that $A$ is an $m \times n$ matrix. If $M$ is obtained from $A$ by an elementary row operation, then there is an $m \times m$ elementary matrix $E$ such that,

$$M = EA$$

• Interestingly, a convenient way to obtain $E$ is to perform the same row operation on the $m \times m$ identity matrix.
  • See the next few slides for examples.
Row Elementary Matrix

- \( M = EA \)

- A convenient way to obtain \( E \) is to perform the same row operation on the \( m \times m \) identity matrix.

- Suppose that we want to switch the first two rows of \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \).

- Performing the same operation on the \( 3 \times 3 \) identity matrix \( I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) yields \( E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). This is the elementary matrix for the above operation.
Row Elementary Matrix

$M = EA$

• A convenient way to obtain $E$ is to perform the same row operation on the $m \times m$ identity matrix.

• Suppose that we want to switch the multiply the second row of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ by 2.

• Performing the same operation on the $3 \times 3$ identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yields $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This is the elementary matrix for the above operation.
Row Elementary Matrix

- $M = EA$

- A convenient way to obtain $E$ is to perform the same row operation on the $m \times m$ identity matrix.

- Suppose that we want to add up the last two rows of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, and use the result to replace the 3rd row.

- Performing the same operation on the $3 \times 3$ identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yields $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. This is the elementary matrix for the above operation.