1 Scalar and Vector Functions

Recall that a function $f$ takes an input, and yields an output. For example, in $f(t) = t^2 + 2t$, the input is $t$, whereas the output is the real value resulting from the calculation $t^2 + 2t$. We say that $f$ is a scalar function if its output is a real value.

The output of a function can also be a vector. In this case, we refer to the function as a vector function. For instance, consider $f(t) = [t^2, 2t, t^3 - t]$. Its input is $t$. For every fixed $t$, $f(t)$ outputs a 3d vector $[t^2, 2t, t^3 - t]$. We will adopt the convention of using boldfaces to represent vector functions.

An input to a function may consist of multiple parameters. For example, $f(x, y) = x^2 + xy + y^3$ and $f(x, y, z) = [xyz, y^3z + y^2]$. If a scalar function $f$ takes $d$ real values as its input, we say that $f$ is a scalar field in $\mathbb{R}^d$. Similarly, if a vector function $f$ takes $d$ real values as its input, we say that $f$ is a vector field in $\mathbb{R}^d$. For example, the $f(x, y)$ and $f(x, y, z)$ shown earlier are a scalar field in $\mathbb{R}^2$ and a vector field in $\mathbb{R}^3$, respectively.

2 Limits and Continuity of One-Variable Vector Functions

Consider first a scalar function $f(t)$ that takes a single real value $t$ as its input. Recall that its limit at $t_0$ is defined as follows:

**Definition 1.** Suppose that a scalar function $f(t)$ is defined around $t_0$ (but not necessarily at $t_0$). We say that

$$\lim_{t \to t_0} f(t) = v$$

if for any real $\delta > 0$, we can find a real value $\epsilon > 0$ such that $|f(t) - v| < \delta$ for all $t$ satisfying $0 < |t - t_0| < \epsilon$.

Now consider a vector function $f(t)$ that takes a single real value $t$ as its input. Suppose that the output of $f(t)$ is a $d$-dimensional vector. By definition, we can write the output vector in its component form $[x_1(t), x_2(t), ..., x_d(t)]$. Now we extend Definition 1 to vector functions:

**Definition 2.** Suppose that $f(t) = [x_1(t), x_2(t), ..., x_d(t)]$ is defined around $t_0$ (but not necessarily at $t_0$). We say that

$$\lim_{t \to t_0} f(t) = [v_1, v_2, ..., v_d]$$

if it holds for each $i \in [1, d]$ that $\lim_{t \to t_0} x_i(t) = v_i$.

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1This means that there is a $\rho > 0$ such that $f(t)$ is defined for $t$ satisfying $0 < |t - t_0| < \rho$. 

For example, suppose that \( f(t) = [t^2, \sin(t)/t] \). Since \( \lim_{t \to 0} t^2 = 0 \) and \( \lim_{t \to 0} \frac{\sin(t)}{t} = 1 \), we know that \( \lim_{t \to 0} f(t) = [0, 1] \).

**Definition 3.** Suppose that \( f(t) = [x_1(t), x_2(t), ..., x_d(t)] \) is defined around \( t_0 \) and at \( t_0 \). We say that \( f(t) \) is continuous at \( t_0 \) if \( \lim_{t \to t_0} f(t) = f(t_0) \).

For example, \( f(t) = [t^2, \sin(t)/t] \) is not continuous at 0 because the function is undefined at \( t = 0 \). On the other hand, \( f(t) = [t^2, \sqrt{t} + 1] \) is continuous at \( t = 0 \). However, the following function is not continuous at \( t = 0 \):

\[
\begin{cases}
  [t^2, \sqrt{t} + 1] & \text{if } t \neq 0 \\
  [0, 2] & \text{if } t = 0
\end{cases}
\]

This is because \( \lim_{t \to 0} f(t) = [0, 1] \neq f(0) \).

### 3 Derivatives of Vector Functions

Recall that derivatives of scalar functions are defined as follows:

**Definition 4.** Suppose that scalar function \( f(t) \) is defined around \( t_0 \) and at \( t_0 \). If the following limit exists:

\[
\lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}
\]

then we say that

- \( f(t) \) is differentiable at \( t_0 \).
- the above limit, denoted as \( f'(t_0) \), is the derivative of \( f(t) \) at \( t = t_0 \).

We now extend the definition to vectors:

**Definition 5.** Suppose that vector function \( f(t) \) is defined around \( t_0 \) and at \( t_0 \). If the following limit exists:

\[
\lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}
\]

then we say that

- \( f(t) \) is differentiable at \( t_0 \).
- the above limit, denoted as \( f'(t_0) \), is the derivative of \( f(t) \) at \( t = t_0 \).

The next important lemma provides another view of the above definition through components:

**Lemma 1.** Suppose that \( f(t) = [x_1(t), x_2(t), ..., x_d(t)] \) is differentiable at \( t_0 \) such that \( f'(t_0) = [y_1(t_0), y_2(t_0), ..., y_d(t_0)] \). Then, \( y_i(t_0) = x'_i(t_0) \) for each \( i \in [1, d] \).

**Proof.** By definition of vector subtraction:

\[
f(t_0 + \Delta t) - f(t_0) = [x_1(t_0 + \Delta t) - x_1(t_0), x_2(t_0 + \Delta t) - x_2(t_0), ..., x_d(t_0 + \Delta t) - x_d(t_0)].
\]
Next, we will prove Rules 1 and 2 in full. The proof for Rule 3 is very tedious but not difficult; we know
\[
\lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t} = [y_1(t_0), y_2(t_0), \ldots, y_d(t_0)]
\] (1)
we know
\[
\lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \left[ \begin{array}{c}
  x_1(t_0 + \Delta t) - x_1(t_0) \\
  x_2(t_0 + \Delta t) - x_2(t_0) \\
  \vdots \\
  x_d(t_0 + \Delta t) - x_d(t_0)
\end{array} \right]
\]
\[
\text{(scalar multiplication)} = \lim_{\Delta t \to 0} \left[ \begin{array}{c}
  \frac{x_1(t_0 + \Delta t) - x_1(t_0)}{\Delta t} \\
  \frac{x_2(t_0 + \Delta t) - x_2(t_0)}{\Delta t} \\
  \vdots \\
  \frac{x_d(t_0 + \Delta t) - x_d(t_0)}{\Delta t}
\end{array} \right]
\]
(from (1)) = \([y_1(t_0), y_2(t_0), \ldots, y_d(t_0)]\).

It thus follows from Definition 2 that, for each \(i \in [1, d]\):
\[
\lim_{\Delta t \to 0} \frac{x_i(t_0 + \Delta t) - x_i(t_0)}{\Delta t} = y_i(t_0).
\]
The left hand side of the above is precisely \(x_i'(t_0)\) by Definition 4. We thus complete the proof. □

The above lemma provides a convenient and intuitive way to compute the derivative of a vector function. For example, consider \(f(t) = [\sin^2 t, \cos^2 t]\). Then we immediately know \(f'(t) = [2\sin(t)\cos(t), -2\sin(t)\cos(t)]\).

Vector derivatives obey some rules that are reminiscent of the corresponding rules on scalar functions:

1. \((f(t) + g(t))' = f'(t) + g'(t)\).
2. \((f(t) \cdot g(t))' = f'(t) \cdot g(t) + f(t) \cdot g'(t)\).
3. Suppose that the outputs of \(f(t)\) and \(g(t)\) are 3d vectors. Then, \((f(t) \times g(t))' = f'(t) \times g(t) + f(t) \times g'(t)\).

Next, we will prove Rules 1 and 2 in full. The proof for Rule 3 is very tedious but not difficult; we will outline its main ideas.

**Proof of Rule 1.** Let \(f(t) = [x_1(t), \ldots, x_d(t)]\) and \(g(t) = [y_1(t), \ldots, y_d(t)]\). From Lemma 1, we know that \(f'(t) = [x_1'(t), \ldots, x_d'(t)]\) and \(g'(t) = [y_1'(t), \ldots, y_d'(t)]\). We have:
\[
(f(t) + g(t))' = [x_1(t) + y_1(t), \ldots, x_d(t) + y_d(t)]'
\]
(by Lemma 1) = \([x_1'(t) + y_1'(t), \ldots, x_d'(t) + y_d'(t)]\)
\[
= f'(t) + g'(t).
\]
□
Proof of Rule 2. Let \( f(t) = [x_1(t), \ldots, x_d(t)] \) and \( g(t) = [y_1(t), \ldots, y_d(t)] \). From Lemma 1, we know that \( f'(t) = [x'_1(t), \ldots, x'_d(t)] \) and \( g'(t) = [y'_1(t), \ldots, y'_d(t)] \). We have:

\[
(f(t) \cdot g(t))' = \left( \sum_{i=1}^{d} x_i(t) \cdot y_i(t) \right)'
= \sum_{i=1}^{d} \left( x'_i(t) \cdot y_i(t) + x_i(t) \cdot y'_i(t) \right)
= \sum_{i=1}^{d} x'_i(t) \cdot y_i(t) + \sum_{i=1}^{d} x_i(t) \cdot y'_i(t)
= f'(t) \cdot g(t) + f(t) \cdot g'(t).
\]