1 Determinant Definition

Let \( A = [a_{ij}] \) be an \( n \times n \) matrix (i.e., \( A \) is a square matrix). Given a pair of \((i, j)\), we define \( A_{ij} \) to be the \((n - 1) \times (n - 1)\) matrix obtained by removing the \( i \)-th row and \( j \)-th column of \( A \). For example, suppose that

\[
A = \begin{bmatrix}
1 & 2 & 1 \\
3 & 0 & -2 \\
-1 & -1 & 2
\end{bmatrix}
\]

Then:

\[
A_{21} = \begin{bmatrix}
2 & 1 \\
-1 & 2
\end{bmatrix}, \quad A_{22} = \begin{bmatrix}
1 & 1 \\
-1 & 2
\end{bmatrix}, \quad A_{32} = \begin{bmatrix}
1 & 1 \\
3 & -2
\end{bmatrix}
\]

We are now ready to define determinants:

**Definition 1.** Let \( A = [a_{ij}] \) be an \( n \times n \) matrix. If \( n = 1 \), its determinant, denoted as \( \text{det}(A) \), equals \( a_{11} \). If \( n > 1 \), we first choose an arbitrary \( i^* \in [1, n] \), and then define the determinant of \( A \) recursively as:

\[
\text{det}(A) = \sum_{j=1}^{n} (-1)^{i^*+j} \cdot a_{i^*j} \cdot \text{det}(A_{i^*j}). \tag{1}
\]

Besides \( \text{det}(A) \), we may also denote the determinant of \( A \) as \(|A|\). Henceforth, if we apply (1) to compute \( \text{det}(A) \), we say that we expand \( A \) by row \( i^* \). It is important to note that the value of \( \text{det}(A) \) does not depend on the choice of \( i^* \). We omit the proof of this fact, but illustrate it in the following examples.

**Example 1 (Second-Order Determinants).** In general, if \( A = [a_{ij}] \) is a \( 2 \times 2 \) matrix, then

\[
\text{det}(A) = a_{11}a_{22} - a_{12}a_{21}.
\]

For instance:

\[
\begin{vmatrix}
2 & 1 \\
-1 & 2
\end{vmatrix} = 2 \times 2 - 1 \times (-1) = 5.
\]

We may verify the above by definition as follows. Choosing \( i^* = 1 \), we get:

\[
\begin{vmatrix}
2 & 1 \\
-1 & 2
\end{vmatrix} = (-1)^{1+1} \cdot 2 \cdot \text{det}(A_{11}) + (-1)^{1+2} \cdot 1 \cdot \text{det}(A_{12})
\]

\[
= 2 \times 2 + (-1) \times (-1) = 5.
\]
Alternatively, choosing $i^* = 2$, we get:

\[
\begin{vmatrix}
2 & 1 \\
-1 & 2
\end{vmatrix} = (-1)^{2+1} \cdot (-1) \cdot \det(A_{21}) + (-1)^{2+2} \cdot 2 \cdot \det(A_{22})
\]

\[= 1 \times 1 + 2 \times 2 = 5.\]

\[\square\]

**Example 2 (Third-Order Determinants).** Suppose that

\[A = \begin{bmatrix}
1 & 2 & 1 \\
3 & 0 & -2 \\
-1 & -1 & 2
\end{bmatrix}\]

Choosing $i^* = 1$, we get:

\[
\det(A) = 1 \begin{vmatrix}
0 & -2 \\
-1 & 2
\end{vmatrix} - 2 \begin{vmatrix}
3 & -2 \\
-1 & 2
\end{vmatrix} + 1 \begin{vmatrix}
3 & 0 \\
-1 & -1
\end{vmatrix}
\]

\[= 1(0 - 2) - 2(6 - 2) + 1(-3 - 0) = -13.\]

Alternatively, choosing $i^* = 2$, we get:

\[
\det(A) = -3 \begin{vmatrix}
2 & 1 \\
-1 & 2
\end{vmatrix} + 0 \begin{vmatrix}
1 & 1 \\
-1 & 2
\end{vmatrix} - (-2) \begin{vmatrix}
1 & 2 \\
-1 & -1
\end{vmatrix}
\]

\[= (-3)(4 + 1) + 0(2 + 1) + 2(-1 + 2) = -13.\]

\[\square\]

### 2 Properties of Determinants

**Expansion by a Column.** Definition 1 allows us to compute the determinant of a matrix by row expansion. We may also achieve the same purpose by column expansion.

**Lemma 1.** Let $A = [a_{ij}]$ be an $n \times n$ matrix with $n > 1$. Choose an arbitrary $j^* \in [1, n]$. The determinant of $A$ equals:

\[
\det(A) = \sum_{i=1}^{n} (-1)^{i+j^*} \cdot a_{ij^*} \cdot \det(A_{ij^*}).
\]

The value of the above equation does not depend on the choice of $j^*$.

We omit the proof but illustrate the lemma with an example below. Henceforth, if we compute $\det(A)$ by the above lemma, we say that we expand $A$ by column $j^*$.

**Example 3.** Suppose that

\[A = \begin{bmatrix}
1 & 2 & 1 \\
3 & 0 & -2 \\
-1 & -1 & 2
\end{bmatrix}\]
Choosing \( j^* = 1 \), we get:

\[
\begin{vmatrix}
0 & -2 & -3 \\
-1 & 2 & 1 \\
-1 & 2 & (-1)
\end{vmatrix} + \begin{vmatrix}
2 & 1 \\
0 & -2
\end{vmatrix} = 0 - 3(4 + 1) - 1(-4 - 0) = -13.
\]

\[\Box\]

**Corollary 1.** Let \( A \) be a square matrix. Then, \( \det(A) = \det(A^T) \).

*Proof.* Note that expanding \( A \) by column \( k \) is equivalent to expanding \( A^T \) by row \( k \).

\[\Box\]

**Corollary 2.** If \( A \) has a zero row or a zero column, then \( \det(A) = 0 \).

*Proof.* If \( A \) has a zero row, then \( \det(A) = 0 \) follows from expanding \( A \) by that row. The case where \( A \) has a zero column is similar.

\[\Box\]

**Determinant of a Row-Echelon Matrix.** The next lemma shows that the determinant of a matrix in row-echelon form is simply the product of the values on the main diagonal.

**Lemma 2.** Let \( A = [a_{ij}] \) be an \( n \times n \) matrix in row-echelon form. Then, \( \det(A) = \prod_{i=1}^{n} a_{ii} \).

*Proof.* We can prove the lemma by induction. First, correctness is obvious for \( n = 1 \). Assuming correctness for \( n \leq t - 1 \) (for \( t \geq 2 \)), consider \( n = t \). Expanding \( A \) by the first row gives:

\[
\det(A) = \sum_{j=1}^{n} (-1)^{1+j} \cdot a_{1j} \cdot \det(A_{1j}). \tag{2}
\]

From induction we know that \( \det(A_{11}) = \prod_{i=2}^{n} a_{ii} \). Furthermore, for \( j > 1 \), \( \det(A_{1j}) = 0 \) because the first column of \( A_{1j} \) contains all 0’s. It thus follows that (2) equals \( \prod_{i=1}^{n} a_{ii} \).

\[\Box\]

**Determinants under Elementary Row Operations.** Let \( A = [a_{ij}] \) be an \( n \times n \) matrix. Recalled that the elementary row operations on \( A \) are:

1. Switch two rows of \( A \).
2. Multiply all numbers of a row of \( A \) by the same non-zero value \( c \).
3. Let \( r_i \) and \( r_j \) be two distinct row vectors of \( A \). Update row \( r_i \) to \( r_i + r_j \).

Next, we refer to the above as Operation 1, 2, and 3, respectively.

**Lemma 3.** The determinant of \( A \)

1. should be multiplied by \(-1\) after Operation 1;
2. should be multiplied by \( c \) after Operation 2;
3. has no change after Operation 3.
Proof. See appendix.

The following corollary will be very useful:

**Corollary 3.** Let $r_i$ and $r_j$ be two distinct row vectors of $A$. The determinant of $A$ does not change after the following operation:

- Update row $r_i$ to $r_i + c \cdot r_j$, where $c$ is a real value.

**Proof.** We consider only $c \neq 0$ (the case of $c = 0$ is trivial). Let $A'$ be the array after applying the above operation. We can also obtain $A'$ by performing the next three operations:

1. Multiply the $j$-th row of $A$ by $c$. Let $A_1$ be the array obtained.
2. Add the $j$-th row of $A_1$ into its $i$-th row. Let $A_2$ be the array obtained.
3. Multiply the $j$-th row of $A_2$ by $1/c$. Let $A_3$ be the array obtained. Note that $A_3 = A'$.

By Lemma 3, $\det(A_1) = c \cdot \det(A)$, $\det(A_2) = \det(A_1)$, and $\det(A_3) = (1/c) \cdot \det(A_3)$. Hence, $\det(A) = \det(A')$.

Let us illustrate Lemma 3 and Corollary 3 with an example.

**Example 4.**

\[
\begin{vmatrix}
1 & 2 & 1 \\
3 & 0 & -2 \\
-1 & -1 & 2 \\
\end{vmatrix} = \begin{vmatrix}
1 & 2 & 1 \\
0 & -6 & -5 \\
0 & 1 & 3 \\
\end{vmatrix} = \begin{vmatrix}
1 & 2 & 1 \\
0 & -6 & -5 \\
0 & 0 & 13/6 \\
\end{vmatrix} = -13.
\]

Here is another derivation giving the same result:

\[
\begin{vmatrix}
1 & 2 & 1 \\
3 & 0 & -2 \\
-1 & -1 & 2 \\
\end{vmatrix} = \begin{vmatrix}
1 & 2 & 1 \\
-1 & -1 & 2 \\
3 & 0 & -2 \\
\end{vmatrix} = \begin{vmatrix}
1 & 2 & 1 \\
0 & 1 & 3 \\
0 & 0 & 13 \\
\end{vmatrix} = -13.
\]

**Corollary 4.** If $A$ has two identical rows or columns, then $\det(A) = 0$.

**Proof.** We prove only the row case. Switching the two rows gets back the same matrix. However, by Lemma 3, the determinant of the matrix should be multiplied by $-1$. Therefore, we get $\det(A) = -\det(A)$, meaning $\det(A) = 0$.

**Determinant under Matrix Multiplication.** The following is a perhaps surprising property of determinants:

**Lemma 4.** Let $A, B$ be $n \times n$ matrices. It holds that $\det(AB) = \det(A) \cdot \det(B)$. 

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The proof is not required, but we will discuss it in a tutorial after we have learned the concept of “matrix inversion”.

Example 5.

\[
\begin{vmatrix}
1 & 2 & 1 \\
3 & 0 & -2 \\
-1 & -1 & 2 \\
\end{vmatrix} = -13
\]

\[
\begin{vmatrix}
-2 & 1 & 0 \\
0 & 1 & 1 \\
1 & -2 & 0 \\
\end{vmatrix} = -3
\]

\[
\begin{bmatrix}
1 & 2 & 1 \\
3 & 0 & -2 \\
-1 & -1 & 2 \\
\end{bmatrix} \begin{bmatrix}
-2 & 1 & 0 \\
0 & 1 & 1 \\
1 & -2 & 0 \\
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 2 \\
-8 & 7 & 0 \\
4 & -6 & -1 \\
\end{bmatrix} = 39.
\]

Relationships with Ranks. The lemma below relates determinants to ranks:

**Lemma 5.** Let \( A \) be an \( n \times n \) matrix. \( A \) has rank \( n \) if and only if \( \det(A) \neq 0 \).

*Proof.* We can first apply elementary row operations to convert \( A \) into row-echelon form \( A^* \). Thus, \( A \) has rank \( n \) if and only if \( A^* \) has rank \( n \). Since \( A^* \) is a square matrix, that it has rank \( n \) is equivalent to saying that all the numbers on its main diagonal are non-zero. Thus, by Lemma 2, we know that \( A^* \) has rank \( n \) if and only if \( \det(A^*) \neq 0 \). Finally, by Lemma 3, \( \det(A) \neq 0 \) if and only if \( \det(A^*) \neq 0 \). We thus complete the proof.

Appendix: Proof of Lemma 3

The claims on Operations 1 and 2 are easy to prove; we leave the proofs to you as exercises.

To prove the claim on Operation 3, we will leverage Corollary 4 (which holds as long as the claim on Operation 1 is true).

Suppose that, after performing Operation 3 on \( A \), we obtain a matrix \( A' \). Our goal is to show that \( \det(A) = \det(A') \). Let us define a new matrix \( B \):

- \( B \) is the same as \( A \), except that the \( i \)-th row of \( B \) is replaced by the \( j \)-th row of \( A \).

In other words, the \( i \)-th row of \( B \) is identical to the \( j \)-th row of \( B \). Corollary 4 tells us that \( \det(B) = 0 \). Next, we will focus on showing:

\[
\det(A') = \det(A) + \det(B)
\]

which will indicate \( \det(A) = \det(A') \) and hence will complete the proof.

Define \( a'_{ik} \) as the number at the \( i \)-th row and \( j \)-th column of \( A' \), and define \( a_{ik}, b_{ik} \) similarly with respect to \( A, B \), respectively. Note that:

\[
a'_{ik} = a_{ik} + b_{ik}
\]
holds by the way $A'$ and $B$ were obtained.

In fact, (3) follows almost directly from the definition of determinants. Let us calculate $\det(A')$ by expanding the matrix on row $i$:

$$
\det(A') = \sum_{k=1}^{n} (-1)^{i+k} a'_{ik} \cdot \det(A'_{ik})
$$

$$
= \sum_{k=1}^{n} (-1)^{i+k} (a_{ik} + b_{ik}) \cdot \det(A'_{ik})
$$

$$
= \left( \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \cdot \det(A'_{ik}) \right) + \left( \sum_{k=1}^{n} (-1)^{i+k} b_{ik} \cdot \det(A'_{ik}) \right)
$$

$$
= \left( \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \cdot \det(A_{ik}) \right) + \left( \sum_{k=1}^{n} (-1)^{i+k} b_{ik} \cdot \det(B_{ik}) \right)
$$

$$
= \det(A) + \det(B).
$$