1 Definition of Arc Lengths

Recall that a curve in $\mathbb{R}^d$ can be represented as a vector function $\mathbf{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$, where $x_1(t), x_2(t), ..., x_d(t)$ give the coordinates of the point on the curve corresponding to a value of $t$. If we take a continuous portion of the curve, we get an arc, which is formally defined as:

**Definition 1.** Given a curve $\mathbf{r}(t)$, an arc of the curve is $\{\mathbf{r}(t) \mid t_0 \leq t \leq t_1\}$ where $t_0$ and $t_1$ are real values.

It is worth mentioning that the arc as defined above is sometimes also referred to as “the curve from $t_0$ to $t_1$” or as “the curve from point $\mathbf{r}(t_0)$ to point $\mathbf{r}(t_1)$”. In the example below, the curve/arc from $t_0$ to $t_1$ is the part of the curve between $p$ and $q$.

Intuitively, an arc should have a “length”, which we formalize below as a limit:

**Definition 2.** Let $C$ be an arc given by $\mathbf{r}(t)$ with $t$ ranging from $t_0$ to $t_1$. Evenly divide the interval $[t_0, t_1]$ by inserting $n + 1$ break points $\tau_0, \tau_1, \tau_2, ..., \tau_n$ where $\tau_0 = t_0$ and $\tau_i - \tau_{i-1} = (t_1 - t_0)/n$ for each $i \in [1, n]$. Define $\sigma_i$ to be the straight line segment connecting the points $\mathbf{r}(\tau_{i-1})$ and $\mathbf{r}(\tau_i)$, and denote by $|\sigma_i|$ the length of $\sigma_i$. Then, if the following limit exists:

$$\lim_{n \to \infty} \sum_{i=1}^{n} |\sigma_i|$$

we say that the limit is the length of $C$. 

![Diagram of an arc and its segments](image)
The figure above shows an example with \( n = 5 \). Note how we approximate the length of the curve by the total length of a sequence of segments.

In this course, we will be interested mainly in smooth curves. Intuitively, these are curves that (i) do not degenerate into a point, and (ii) do not have “corners” (e.g., the boundary of a triangle is not smooth). Mathematically, we formalize the notion as follows:

**Definition 3.** Let \( C \) be a curve given by \( \mathbf{r}(t) \) with \( t \) ranging from \( t_0 \) to \( t_1 \). \( C \) is smooth if (i) \( \mathbf{r}'(t) \) is continuous in \([t_0, t_1]\), and (ii) \( \mathbf{r}'(t) \neq \mathbf{0} \) at any \( t \in [t_0, t_1] \).

We will state without proof the following lemma:

**Lemma 1.** Let \( C \) be as described in Definition 2. If \( C \) is smooth, then the limit (1) always exists.

## 2 Computing Arc Lengths

Consider a curve given by the vector function \( \mathbf{r}(t) \). Fix a real value \( t_0 \), and consider the arc \( C \) from \( t_0 \) to \( t \). Note that \( C \) extends as \( t \) grows, which means that the length \( s \) of \( C \) is a function of \( t \).

The following is an important lemma:

**Lemma 2.** If \( C \) is smooth, then it holds that:

\[
\frac{d(s(t))}{dt} = \sqrt{\sum_{i=1}^{d} \left( \frac{d(x_i(t))}{dt} \right)^2}.
\]

We will not present a complete proof of the lemma, but the following discussion will point out the main ideas. Consider the figure below in 2d space. Imagine that we increase \( t \) by a tiny amount \( \Delta t \). By doing so, we have traveled on the curve a little from point \( p \) to point \( q \). \( \Delta x_1 \) and \( \Delta x_2 \) give the coordinate differences of \( p \) and \( q \) on the two dimensions, respectively. When \( \Delta t \) is extremely small, the length \( \Delta s \) of the curve from \( p \) to \( q \) should be very close to the length of the segment connecting \( p \) and \( q \), that is, \( \Delta s \approx \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2} \), which gives \( \frac{\Delta s}{\Delta t} \approx \sqrt{\left( \frac{\Delta x_1}{\Delta t} \right)^2 + \left( \frac{\Delta x_2}{\Delta t} \right)^2} \).

Now fix another real value \( t_1 \geq t_0 \). Denote by \( L \) the length of the arc from \( t_0 \) to \( t_1 \). We can calculate \( L \) as follows:

\[
L = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} \frac{ds}{dt} dt = \int_{t_0}^{t_1} \sqrt{\sum_{i=1}^{d} \left( \frac{d(x_i(t))}{dt} \right)^2} dt.
\]
Example 1. Consider the circle \( x^2 + y^2 = 1 \). Let \( p \) be the point \((1, 0)\) and \( q \) the point \((-1, 0)\). Let \( C \) be the arc of the circle from \( p \) to \( q \). How to calculate the length of \( C \)?

First of all, we need to represent the circle using a single parameter. One way of doing so is to define:

\[
\begin{align*}
  x(t) &= \cos(t) \\
  y(t) &= \sin(t).
\end{align*}
\]

Then \( C \) is essentially the curve from \( t = 0 \) (point \( p \)) to \( t = \pi \) (point \( q \)). Hence, the length of \( C \) is given by:

\[
\begin{align*}
  \int_0^\pi ds dt &= \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
  &= \int_0^\pi \sqrt{(-\sin(t))^2 + (\cos(t))^2} dt \\
  &= \int_0^\pi 1 dt = \pi.
\end{align*}
\]

Example 2. Consider the helix \( \mathbf{r}(t) = [x(t), y(t), z(t)] \) where

\[
\begin{align*}
  x(t) &= \cos(t) \\
  y(t) &= \sin(t) \\
  z(t) &= t.
\end{align*}
\]

The length of the arc from \( t = 0 \) to \( t = \pi \) is:

\[
\begin{align*}
  \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt &= \int_0^\pi \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2} dt \\
  &= \sqrt{2} \int_0^\pi dt \\
  &= \sqrt{2} \pi.
\end{align*}
\]

\[\square\]