Exercises: Similarity Transformation

Problem 1. Diagonalize the following matrix:

\[
A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}
\]

Solution. Matrix \( A \) has two eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = 2 \). Since (i) \( A \) is a \( 2 \times 2 \) matrix and (ii) it has 2 distinct eigenvalues, we can apply the diagonalization method we discussed in class. Specifically, we obtain an arbitrary eigenvector \( \mathbf{v}_1 \) of \( \lambda_1 \), say \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \) and, and an arbitrary eigenvector \( \mathbf{v}_2 \) of \( \lambda_2 \), say \( \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). Then, we form:

\[
Q = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}
\]

by using \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) as the first and second columns, respectively. \( Q \) has the inverse:

\[
Q^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}
\]

We thus obtain the following diagonalization of \( A \):

\[
A = Q \text{ diag}[3, 2] Q^{-1}.
\]

Problem 2. Consider again the matrix \( A \) in Problem 5. Calculate \( A^t \) for any integer \( t \geq 1 \).

Solution. We already know that \( A \):

\[
A = Q \text{ diag}[3, 2] Q^{-1}.
\]

Hence:

\[
A^t = Q \text{ diag}[3^t, 2^t] Q^{-1}
\]

\[
= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 2^t \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} -3^t + 2^{t+1} & -3^t + 2^t \\ 2 \times 3^t - 2^{t+1} & 2 \times 3^t - 2^t \end{bmatrix}
\]

Problem 3. Diagonalize the matrix \( A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \).

Solution. Recall that all symmetric matrices are diagonalizable. \( A \) is a \( 3 \times 3 \) matrix. The key is to find three linearly independent eigenvectors.
From the solution of Problem 1, we know that $A$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. 

$\text{EigenSpace}(\lambda_1)$ includes all \[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]
satisfying
\[
\begin{align*}
x_1 &= u \\
x_2 &= v \\
x_3 &= u
\end{align*}
\]
for any $u, v \in \mathbb{R}$. The vector space $\text{EigenSpace}(\lambda_1)$ has dimension 2 with a basis $\{v_1, v_2\}$ where $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (given by $u = 1, v = 0$) and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (given by $u = 0, v = 1$).

Similarly, $\text{EigenSpace}(\lambda_2)$ includes all \[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]
satisfying
\[
\begin{align*}
x_1 &= u \\
x_2 &= 0 \\
x_3 &= -u
\end{align*}
\]
for any $u \in \mathbb{R}$. The vector space $\text{EigenSpace}(\lambda_2)$ has dimension 1 with a basis $\{v_3\}$ where $v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (given by $u = 1$).

So far, we have obtained three linearly independent eigenvectors $v_1, v_2, v_3$ of $A$. We can then apply the diagonalization method exemplified in Problem 5 to diagonalize $A$. Specifically, we form:

$$ Q = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} $$

$Q$ has the inverse:

$$ Q^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix} $$

We thus obtain the following diagonalization of $A$:

$$ A = Q \text{diag}[1, 1, -1] Q^{-1}. $$

**Problem 4.** Suppose that matrices $A$ and $B$ are similar to each other, namely, there exists $P$ such that $A = P^{-1}BP$. Prove: if $x$ is an eigenvector of $A$ under eigenvalue $\lambda$, then $Px$ is an eigenvector of $B$ under eigenvalue $\lambda$.

**Solution.** By definition of similarity, we know $A = P^{-1}BP$. We proved in the lecture that $\lambda$ must also be an eigenvalue of $B$. Since $x$ is an eigenvector of $A$ under $\lambda$, we know:

$$ Ax = \lambda x \Rightarrow $$

$$ P^{-1}BPx = \lambda x \Rightarrow $$

$$ B(Px) = \lambda(Px) $$
which completes the proof.

**Problem 5.** Suppose that an $n \times n$ matrix $A$ has $n$ linearly independent eigenvectors $v_1, v_2, \ldots, v_n$. Prove: for any $n \times 1$ vector $x$, $Ax$ is a linear combination of $v_1, v_2, \ldots, v_n$.

**Solution.** Assume that $v_i$ ($i \in [1, k]$) is an eigenvector of $A$ under eigenvalue $\lambda_i$. We have $Av_i = \lambda_i v_i$. Since $v_1, v_2, \ldots, v_n$ are linearly independent, we know that $x$ must be a linear combination $v_1, v_2, \ldots, v_n$. Namely, there exist $c_1, \ldots, c_n$ such that

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \Rightarrow$$

$$Ax = c_1 Av_1 + c_2 Av_2 + \ldots + c_n Av_n \Rightarrow$$

$$Ax = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n,$$

which completes the proof.

**Problem 6.** Prove or disprove: if an $n \times n$ matrix $A$ has rank $n$, then it must have $n$ independent eigenvectors.

**Solution.** False. Consider $n = 2$ and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has only one distinct eigenvalue 1. Thus, any eigenvector $v$ of $A$ must satisfy:

$$(A - I)x = 0 \Rightarrow$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = 0$$

Thus, any eigenvector of $A$ must have the form $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}$. This set of vectors has a dimension of 1.

**Problem 7.** Prove that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

**Solution.** $A$ has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Let $v_1$ be an eigenvector of $\lambda_1$. $v_1$ must satisfy:

$$(A - \lambda_1 I)v_1 = 0 \Rightarrow$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v_1 = 0 \Rightarrow$$

Hence, the set of eigenvectors of $\lambda_1$ is:

$$\left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}$$

This set has dimension 1.
Let \( v_2 \) be an eigenvector of \( \lambda_2 \). \( v_2 \) must satisfy:

\[
(A - \lambda_1 I)v_2 = 0 \Rightarrow \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix} v_2 = 0 \Rightarrow
\]

Hence, the set of eigenvectors of \( \lambda_2 \) is:

\[
\left\{ \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}
\]

This set also has dimension 1.

It thus follows that the largest number of linearly independent eigenvectors of \( A \) is \( 1 + 1 = 2 \). Therefore, \( A \) is not diagonalizable.

**Problem 8.** Let \( A, B, \) and \( C \) be three \( n \times n \) matrices for some integer \( n \). Prove that if \( A \) is similar to \( B \) and \( B \) is similar to \( C \), then \( A \) is similar to \( C \).

**Solution.** From the fact that \( A \) is similar to \( B \) and \( B \) is similar to \( C \), we know:

\[
A = P^{-1}BP
\]

and

\[
B = Q^{-1}CQ.
\]

Hence:

\[
A = P^{-1}Q^{-1}BQP = (QP)^{-1}B(QP)
\]

which completes the proof.

**Problem 9.** Decide whether

\[
A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}
\]

is similar to

\[
B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.
\]

**Solution 1.** From Problem 1, we know that \( A \) has distinct eigenvalues 3 and 2. Hence, \( A \) is similar to the diagonal matrix \( diag[3, 2] \). On the other hand, \( B \) clearly also has eigenvalues 3 and 2, and thus, is also similar to \( diag[3, 2] \). From the result of Problem 8, we know that \( A \) is similar to \( B \).
Solution 2. We will try to find an invertible matrix \( P = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \) that makes \( A = PB P^{-1} \) hold. This is equivalent to \( AP = PB \). Hence:

\[
\begin{bmatrix}
1 & -1 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
x & y \\
z & w
\end{bmatrix}
= \begin{bmatrix}
x & y \\
z & w
\end{bmatrix}
\begin{bmatrix}
3 & 1 \\
0 & 2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
x - z & y - w \\
x + 4z & 2y + 4w
\end{bmatrix}
= \begin{bmatrix}
3x & x + 2y \\
3z & z + 2w
\end{bmatrix}
\]

This gives the following equation set:

\[
\begin{align*}
x - z &= 3x \\
y - w &= x + 2y \\
2x + 4z &= 3z \\
2y + 4w &= z + 2w
\end{align*}
\]

You can verify that the set of solutions \( \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \) is

\[
\begin{bmatrix}
-u/2 \\
u/2 - v \\
u \\
v
\end{bmatrix}
\quad| \quad u \in \mathbb{R}, v \in \mathbb{R}
\]

Let us try \( u = 2, v = 0 \). This gives \( P = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix} \). Since \( det(P) \neq 0 \), we know that \( P \) is invertible. We can now conclude that \( A \) is similar to \( B \).